The Signature Theorem

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Oriented cobordism ring

The signature and the L genus

Sketch of the proof of the signature theorem

Oriented cobordant classes

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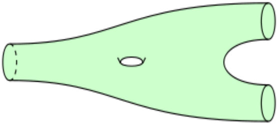
Def. Two smooth compact oriented n-dimensional manifolds M and M' are said to be **oriented cobordant** if there exists a smooth, compact and oriented manifold-with-boundary X such that ∂X with its induced orientations is diffeomorphic to $M + (-M')$.



The oriented cobordism ring Ω_*

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Def. The set Ω_* consisting of all oriented cobordism classes of n-dimensional manifolds forms an Abelian group under $+$. Furthermore, the cartesian product gives rise to an associative bilinear product operation $\Omega_n \times \Omega_m \longrightarrow \Omega_{n+m}$.
Thus, the sequence

$$\Omega_* = (\Omega_0, \Omega_1, \Omega_2, \cdots)$$

of oriented cobordism groups has a structure of a commutative graded ring.

Examples

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- $\Omega_1 = 0$.
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- $\Omega_8 \cong \mathbb{Z} \oplus \mathbb{Z}$. Is generated by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$.

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Theorem (Thom)

The tensor product $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ with independent generators $\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \cdots$

Definition of the signature

- Let us consider an oriented, compact smooth manifold without boundary M of dimension $4k$. The cup product in cohomology at level $2k$ defines a symmetric quadratic form

$$H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \xrightarrow{\cup} H^{4k}(M;\mathbb{Q}) \xrightarrow{\mu_M} \mathbb{Q}$$

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- By Poincaré duality this is a non-degenerate quadratic form. We define the **signature** $\sigma(M)$ to be the signature of this quadratic form. This means that if $a_1, \cdots, a_r$ is a basis for $H^{2k}(M;\mathbb{Q})$ so that the symmetric matrix

$$[\langle a_i \cup a_j, \mu_M \rangle]_{ij}$$

is diagonal, then σ_M is equal to the number of positive diagonal entries minus the number of negative ones.

Remarks

- Via the de Rham theorem we can could define the signature via differential forms.

$$H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \xrightarrow{\wedge} H^{4k}(M;\mathbb{Q}) \xrightarrow{\int_M} \mathbb{Q}$$

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Example Let us consider the complex projective space $\mathbb{C}P^{2k}$. Its cohomology ring is

$$H^{\bullet}(\mathbb{C}P^{2k};\mathbb{Q}) \cong \mathbb{Q}[a]/a^{2k+1}$$

In particular $H^{2k}(\mathbb{C}P^{2k};\mathbb{Q})$ is generated by a single element a^k with

$$\langle a^k \cup a^k, \mu_{\mathbb{C}P^{2k}} \rangle = 1$$

Hence $\sigma(\mathbb{C}P^{2k}) = 1$.

The signature as a ring homomorphism

Theorem (Thom)

The signature satisfies

- $\sigma(M + M') = \sigma(M) + \sigma(M')$
- $\sigma(M \times M') = \sigma(M)\sigma(M')$
- *If M is an oriented boundary, then $\sigma(M) = 0$*
- $\sigma(\mathbb{C}P^{2k}) = 1$

Thus, $\sigma : \Omega_* \longrightarrow \mathbb{Z}$ is the unique ring homomorphism taking the value 1 on each $\mathbb{C}P^{2k}$.

Multiplicative Sequences

- Let $\mathbb{Q}[[x]]^*$ be the multiplicative group of formal power series with rational coefficients and constant term 1. Fix an element $f(x) \in \mathbb{Q}[[x]]^*$ and for each $n \in \mathbb{N}$ consider the formal power series in n variables given by $f(x_1)f(x_2)\cdots f(x_n)$.

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- Since this expression in symmetric in the x_j's, it has an expansion of the form

$$f(x_1)f(x_2)\cdots f(x_n) = 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + \cdots$$

  where each σ_k is the elementary symmetric polynomial of degree k and each F_k is weighted homogeneous of degree k, i.e.,

$$F_k(t\sigma_1, \cdots, t^k\sigma_k) = t^k F_k(\sigma_1, \cdots, \sigma_k)$$

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Let B be a $\mathbb{Z}$-graded commutative algebra $B = B_0 \oplus B_1 \oplus \cdots$ and let B^* be the multiplicative group consisting of the elements of the form $b = 1 + b_1 + b_2 + \cdots$, such that $b_k \in B_k$.

Theorem

Fix a multiplicative sequence $\{F_k(\sigma_1, \cdots, \sigma_k)\}_{k=1}^{\infty}$ and let B^* as above. Define a map $\mathbf{F} : B^* \longrightarrow B^*$ by

$$\mathbf{F}(b) = 1 + F_1(b_1) + F_2(b_1, b_2) + \cdots$$

Then $\mathbf{F}$ is a group homomorphism, i.e.,

$$\mathbf{F}(bc) = \mathbf{F}(b)\mathbf{F}(c)$$

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Pontrjagin genus

Def. To each smooth manifold M we associate the total $\mathbf{F}$-class (or Pontrjagin genus)

$$\mathbf{F}(M) = \mathbf{F}(p(TM))$$

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Thm. If M is compact oriented and of dimension n with fundamental class μ_M, the map

$$F(M) = \langle \mathbf{F}(M), \mu_M \rangle$$

defines a ring homomorphism

$$F : \Omega_* \longrightarrow \mathbb{Q}$$

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The signature Theorem

Theorem (Hirzebruch)

Let M be a compact, closed and oriented smooth manifold of dimension $4k$. Then

$$\sigma(M) = L(M)$$

where $L(M)$ is the Pontrjagin genus associated to the holomorphic function

$$\ell(x) = \sqrt{x}/\tanh\sqrt{x} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + \cdots$$

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For example, the first terms are

- $L_1 = \frac{1}{3}p_1$
- $L_2 = \frac{1}{45}(7p_2 - p_1^2)$

Sketch of the proof of the signature theorem

- Since the correspondences $M \mapsto \sigma(M)$ and $M \mapsto L(M)$ both give rise to ring homomorphisms $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$, it suffices to verify the theorem for the generators of the algebra $\Omega_* \otimes \mathbb{Q}$, which we know that are the spaces $\mathbb{C}P^{2k}$.

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- Hence, $L(M)$ is just the coefficient of a^{2k} in this power series.

Sketch of the proof of the signature theorem

- If we go to complex variable $a \mapsto z$, the coefficient c_{2k} of z^{2k} in the Taylor expansion $(z/\tanh z)^{2k+1}$ can be computed as

$$c_{2k} = \oint_{S^1} \frac{1}{2\pi i z^{2k+1}} \left(\frac{z}{\tanh z}\right)^{2k+1} dz$$

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- Therefore

$$c_{2k} = \frac{1}{2\pi i} \oint \frac{(1 + u^2 + u^4 + \cdots)}{u^{2k+1}} du = 1$$