INTRODUCTION TO THE MOMENT MAP

JUAN CAMILO ORDUZ

ABSTRACT. This short notes were a guide for a short communication given in the *Summer School on Geometrical, Algebraic and Topological Methods on Quantum Field Theory*, Villa de Leyva, Colombia, 2011.

CONTENTS

1. SYMPLECTIC GEOMETRY

A good introduction to symplectic geometry is [2] and a very complete reference on symplectic geometry and classical mechanics is [1].

Definition 1.1. Let M be a smooth manifold and $\omega \in \Omega^2(M)$ be a 2-form on M. Then ω is said to be symplectic if it is closed, i.e. $d\omega = 0$, and if it is non-degenerated.The pair (M, ω) is said to be a **symplectic manifold**.

Remark 1.1. Let $A \in M_n(\mathbb{R})$ be a real skew-symmetric square matrix. Then $\det A = (-1)^n \det(A)$. This argument shows that a symplectic manifold must have even dimension.

Example 1.1 (Phase Space [3])**.** Let Q be a smooth manifold (in the context of classical mechanics this is the **configuration space**), we shall see that the cotangent bundle T^*Q (called **phase space**) has a natural symplectic structure: Let $\alpha \in T^*Q$ and $v \in T_\alpha(T^*Q)$, then we define the **Liouville 1-form** θ on T^*Q by $\langle \theta_\alpha, v \rangle = \langle \alpha, (\pi_* v) \rangle$. Where π_* denotes the pushforward of the projection $\pi : T^*Q \to Q$. (Here $\langle , \rangle$ denotes the pairing on 1-forms and tangent vectors). Finally we can take the exterior derivative to obtain the symplectic form $\omega = d\theta$.
In local coordinates $q^1, ..., q^n, p_1, ..., p_n$ the symplectic form can be written as $\omega = dp_i \wedge dq^i$, where the summation convention for repeated indices is applied.

Date: August 22, 2012.

Notice that if $\phi \in \text{Diff}(Q)$ then clearly $\phi^* \in \text{Diff}(T^*Q)$, but a nice property is that ϕ^* preserves the Liouville 1-form. To prove it let $\psi = \phi^*$ and compute for a vector field v on T^*Q:

$$\begin{aligned}\langle(\psi^*\theta)_\alpha, v_\alpha\rangle &= \langle\theta_{\psi(\alpha)}, \psi_* v_\alpha\rangle = \langle\psi(\alpha), \pi_*\psi_* v\rangle \\ &= \langle\phi^*(\alpha), \phi_*^{-1}\pi_* v_\alpha\rangle = \langle\alpha, \pi_* v_\alpha\rangle \\ &= \langle\theta_\alpha, v_\alpha\rangle\end{aligned}$$

where we have used the fact that $\pi(\phi^*\alpha) = \phi^{-1}(\pi(\alpha))$.

Definition 1.2. Let (M, ω) be a symplectic manifold, and $f \in C^\infty(M)$. Then $X_f \in \mathfrak{X}(M)$ is said to be (globally) **Hamiltonian** with respect to f if

$$i_{X_f}\omega + df = 0.$$

The vector field X_f is also called the Hamiltonian vector field associated to $f \in C^\infty(M)$. The space of Hamiltonian vector fields over M is denoted by HamVF(M).

Example 1.2. Consider the symplectic manifold $(T^*\mathbb{R} \cong \mathbb{R}^2, dp \wedge dq)$ and let H be a real smooth function over $T^*\mathbb{R}$. Then, if we compute

$$(dp \wedge dq)(\partial_p H \partial_q - \partial_q H \partial_p, \cdot) = -\partial_p H dp - \partial_q H dq = -dH,$$

so $X_H = \partial p H \partial_q - \partial_q H \partial_p$.

Remark 1.2. Symplectic geometry arises naturally in classical mechanics since, if H is the Hamiltonian of a classical system, then the equations of motion of a free system are given by $i_X\omega + dH = 0$, where ω is the symplectic form of the phase space of the configuration space mentioned above.

For instance consider the last example: if $\varphi_t^{X_H}$ is the flow of X_H then

$$\frac{d}{dt}\varphi_t^{X_H} = (X_H)_{\varphi_t^{X_H}},$$

which implies that

$$\dot{q}\partial_q + \dot{p}\partial_p = \partial p H \partial_q - \partial_q H \partial_p,$$

from where we can read Hamilton's equations $\dot{q} = \partial p H$ and $\dot{p} = -\partial q H$.

Definition 1.3. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A smooth map $f : M_1 \longrightarrow M_2$ is said to be **symplectic** if $f^*\omega_2 = \omega_1$.

The following is a important theorem [1]:

Theorem 1.1 (Liouville). *Let X_f be a Hamiltonian vector field, and $\varphi_t^{X_f}$ be its flow. Then $\varphi_t^{X_f}$ is symplectic. In particular $L_{X_f}\omega = 0$.*

Moreover, if X is a vector field on M such that ϕ_t^X is symplectic then $L_X\omega = 0$, this implies that $i_X\omega$ is closed since $di_X\omega = (L_X - i_X d)\omega = 0$. Does it imply that X is Hamiltonian? In general no, for instance, if $H^1_{dR}(M)$ does not vanish, it can not always be the case. Nevertheless, X is always *locally* Hamiltonian as a result of Poincaré lemma.

A symplectic structure on a symplectic manifold gives rise to a natural Poisson structure on its algebra of functions $C^\infty(M)$.

Definition 1.4. The map $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined by $\{f,g\} = \omega(X_f, X_g)$ is called the **Poisson Bracket**.

Remark 1.3. Note that

$$\{f,g\} = \omega(X_f, X_g) = -i_{X_f} i_{X_g}\omega = i_{X_f} dg = X_f(g) = L_{X_f} g.$$

This implies that

$$d\{f,g\} = d(L_{X_f} g) = L_{X_f} dg = -L_{X_f} i_{X_g}\omega = -i_{X_g} L_{X_f}\omega - i_{[X_f,X_g]}\omega,$$

hence

$$X_{\{f,g\}} = [X_f, X_g].$$

In addition, the Poisson bracket can be seen as a derivation in the following sense:

$$\{f,gh\} = L_{X_f} gh = gL_{X_f} h + hL_{X_f} g = g\{f,h\} + h\{f,g\}.$$

Proposition 1.1. *The Jacobi identity for the Poisson bracket defined before is equivalent to the fact that the symplectic form ω is closed.*

Proof. Let X_f, X_g and X_h be Hamiltonian vector fields, we compute explicitly (refer to [5] for Cartan's identities)

$$\begin{aligned} d\omega(X_f, X_g, X_h) =& X_f\omega(X_g, X_h) + X_g\omega(X_h, X_g) + X_h\omega(X_f, X_g) \\ &+ \omega(X_f, [X_g, X_h]) + \omega([X_g, X_h], X_f) + \omega(X_h, [X_f, X_g]) \\ =& X_f\omega(X_g, X_h) + X_g\omega(X_h, X_g) + X_h\omega(X_f, X_g). \\ &+ \{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\}. \end{aligned}$$

Using Remark 1.3 we see that

$$\begin{aligned} X_f\omega(X_f, X_g) = X_f(\{g,h\}) = d(\{g,h\})(X_f) = (-i_{[X_g,X_h]}\omega)X_f \\ = \omega(X_f, [X_g, X_h]) = \{f, \{g,h\}\}, \end{aligned}$$

therefore we conclude that

$$d\omega(X_f, X_g, X_h) = 2(\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\}).$$

□

We have proved the following proposition:

Proposition 1.2. *The pair $(C^\infty(M), \{,\})$ has a Lie algebra structure. Moreover, we have a short exact sequence of Lie algebras*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\imath} C^\infty(M) \xrightarrow{\nu} HamVF(M) \longrightarrow 0$$

since if $X_f = X_g$, then f and g must differ by a constant. Here $\nu(f) = X_f$ and HamVF(M) denotes the space of Hamiltonian vector fields on (M,ω).

2. The Moment Map

The main reference for this part is [4].

Definition 2.1. Let M be a smooth manifold. A **left action** of a Lie group G on M is a smooth map $\cdot : G \times M \to M$ such that:

(1) $e \cdot x = x \quad \forall x \in M$ (where e is the identity of G)
(2) $(gh) \cdot x = g \cdot (h \cdot x) \quad \forall g, h \in G \, \forall x \in M$

Definition 2.2. Let us denote by $\mathfrak{g}$ the Lie algebra of G. Fix $A \in \mathfrak{g}$ and $x \in M$. Note that $t \longmapsto \exp(tA) \cdot x$ defines a curve on M. The corresponding vector field defined by

$$\beta(A)(x) = \beta^A(x) = \frac{d}{dt}\bigg|_{t=0} \exp(tA) \cdot x$$

is called the **infinitesimal generator** of the action corresponding to A.

Proposition 2.1. [1] *Let $A, B \in \mathfrak{g}$, then $\beta([A,B]) = -[\beta^A, \beta^B]$. In other words, if we define $\gamma(A) = -\beta(A)$, then $\gamma : \mathfrak{g} \longmapsto \mathfrak{X}(M)$ is a Lie algebra homomorphism.*

We would like to restrict ourselves to an action of a Lie group G on a symplectic manifold (M, ω) such that $\gamma(A)$ is a globally Hamiltonian vector field (**Hamiltonian action**) for each $A \in \mathfrak{g}$. The reason is that we would like to relate classical observables with the elements of $\mathfrak{g}$ via the homomorphism γ. Two common cases would be when $H^1_{dR}(M) = 0$ or when $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

In this context it is natural to ask when does the dotted lift $\hat{J} : \mathfrak{g} \longrightarrow C^\infty(M)$ in the diagram (2.1) exist such that it is a linear Lie algebra homomorphism and makes the diagram commute, that is $\gamma^A = \nu(\hat{J}^A) = X_{\hat{J}^A}$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & C^\infty(M) \arrow[r, "\nu"] & \mathrm{HamVF}(M) \arrow[r] & 0. \\
& & & \mathfrak{g} \arrow[u, "\gamma"'] \arrow[ul, dashed, "\hat{J}"] &
\end{tikzcd} \tag{2.1}$$

It is not difficult to find a linear map that does this job. However the Lie homomorphism requirement would need a little work. For instance, if we compute

$$\nu(\hat{J}([A,B]) = \gamma^{[A,B]} = [\gamma^A, \gamma^B] = [\nu(\hat{J}(A)), \nu(\hat{J}(B))] = \nu(\{\hat{J}(A), \hat{J}(B)\}),$$

we note that the problem may be solved up to a constant. Define $z(A,B) \in \mathbb{R}$ by

$$z(A,B) = \{\hat{J}(A), \hat{J}(B)\} - \hat{J}([A,B]).$$

Can we make this constant vanish for all $A, B \in \mathfrak{g}$? This question and the meaning of this constant will be treated more carefully in the next section.

Suppose that two of such linear maps $\hat{J}, \hat{J}' : \mathfrak{g} \to C^\infty(M, \mathbb{R})$ are given. Then there exists $h \in \mathfrak{g}^*$ such that

$$\hat{J}' = \hat{J} + h.$$

If one computes $\{\hat{J}'(A), \hat{J}'(B)\} = z(A, B) - h([A, B])$, one sees that the problem will be solved if $z(A, B) = h([A, B])$. The existence of such an $h \in \mathfrak{g}^*$ will be an algebraic property of $\mathfrak{g}$.

Proposition 2.2. [3] *Let $A, B, C \in \mathfrak{g}$, then*

(1) $z(A, B) = -z(B, A)$.
(2) $z(A, [B, C]) + z(B, [C, A]) + z(C, [A, B]) = 0$ *(Jacobi Identity).*

Definition 2.3. For a Hamiltonian symplectic action of a Lie group G on a symplectic manifold (M, ω), if there exists a Lie algebra homomorphism $\hat{J} : \mathfrak{g} \longrightarrow C^\infty(M)$ that makes the diagram (2.1) commutes, then the map

$$J : M \longmapsto \mathfrak{g}^*$$

defined by $J(x)(A) = \hat{J}(A)(x)$, for $A \in \mathfrak{g}$, $x \in M$, is called a **moment map** of the action.

Theorem 2.1 (E. Noether [4]). *Consider a Hamiltonian symplectic action of a Lie group G on a symplectic manifold (M, ω) with momentum map J. If $H : M \longmapsto \mathbb{R}$ is a G-invariant function (i.e. $H(x) = H(g \cdot x) \quad \forall g \in G$), then J is a constant of the motion H, that is, $J \circ \varphi_t^{X_H} = J$, where $\varphi_t^{X_H}$ is the flow of X_H.*

Proof. Let us compute first

$$\begin{aligned} \{\hat{J}(A), H\}(x) &= (X_{\hat{J}(A)} H)(x) = (\gamma(A)H)(x) \\ &= \frac{d}{dt}\bigg|_{t=0} H(\exp(tA) \cdot x) = \frac{d}{dt}\bigg|_{t=0} H(x) = 0 \end{aligned}$$

Therefore, $\hat{J}(A)(\varphi_t^{X_H})$ is constant, but $\varphi_0^{X_H}(x) = x$. $\square$

2.1. **Phase Space.** Suppose that we have an action of a Lie group G on the configuration space Q. This action can be lifted to T^*Q using the pullback as we did before. Since the lifted action leaves the Liouville 1-form invariant we see that $L_{\gamma(A)}\theta = 0$ for $A \in \mathfrak{g}$. Therefore

$$L_{\gamma(A)}\theta = (i_{\gamma(A)}d + di_{\gamma(a)})\theta = i_{\gamma(A)}\omega + d(i_{\gamma(A)}\theta) = 0,$$

so it is natural to define $\hat{J}(A) = i_{\gamma(A)}\theta$.

Let us show now that the obstruction vanishes. Note that $\{\hat{J}(A), \hat{J}(B)\} = \omega(\gamma(A), \gamma(B))$. On the other hand

$$\hat{J}([A, B]) = i_{\gamma([A,B])}\theta = i_{[\gamma(A),\gamma(B)]}\theta.$$

Using Cartan's identities and the fact that $L_{\gamma(A)}\theta = 0$ we see that

$$i_{[\gamma(A),\gamma(B)]}\theta = L_{\gamma(a)} i_{\gamma(B)}\theta - i_{\gamma(B)} L_{\gamma(A)}\theta = \gamma(A)(\theta(\gamma(B))).$$

Recall the explicit for the exterior derivate for a 1-form is

$$\begin{aligned}\omega(\gamma(A),\gamma(B)) &= d\theta(\gamma(A),\gamma(B)) \\ &= \gamma(A)(\theta(\gamma(B))) - \gamma(B)(\theta(\gamma(A))) - \theta([\gamma(A),\gamma(B)]).\end{aligned}$$

So we can conclude that

$$\omega(\gamma(A),\gamma(B)) = \gamma(A)(\theta(\gamma(B))),$$

and therefore $\{\hat{J}(A), \hat{J}(B)\} = \hat{J}([A, B])$.

We now want to find an expression for the moment map in terms of the infinitesimal generators $\tilde{\gamma} : \mathfrak{g} \longrightarrow \mathfrak{X}(Q)$ on Q. Note that from the definition of the Liouville 1-form we have, for $\alpha(q, p_q) \in T^*_q M$,

$$\theta_\alpha(\gamma(A)_\alpha) = p_q((\pi_*)_\alpha \gamma(A)_\alpha) = p_q(\tilde{\gamma}(A)_q).$$

Therefore we, can write the moment map in this example by

$$J(\alpha)(A) = p_q(\tilde{\gamma}(A)_q). \tag{2.2}$$

Example 2.1. [3] Consider the configuration space to be $Q = \mathbb{R}^3$, and the rotation group $G = SO(3)$ acting on it. Recall that its Lie algebra is given by $\mathfrak{so}(3) = \{A \in M_3(\mathbb{R}) \,|\, A + A^t = 0\}$, we can define a Lie algebra isomorphism $\iota : \mathfrak{so}(3) \xrightarrow{\cong} \mathbb{R}^3$ by

$$\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

where $\iota([A, B]) = \iota(A) \times \iota(B)$. Now we are going to compute the infinitesimal generator $\tilde{\gamma}(A)$ on Q:

$$\tilde{\gamma}(A)(q) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\iota(A)) \cdot q = \left.\frac{d}{dt}\right|_{t=0} \iota(tA) \cdot q = \iota(A) \times q$$

Therefore, using equation (2.2), we see that $\hat{J}(A)(q, p) = p \cdot (\iota(A) \times q) = \iota(A)(q \times p)$. Hence

$$J(q, p) = p \times q$$

which we interpret as the angular momentum, as we expected.

3. A few words on the obstruction

The following section describes a method developed in [3] to ensure that the map $\hat{J}$ is indeed a Lie algebra homomorphism, nevertheless there is a price to pay: we will need to enlarge the Lie algebra. This method is fundamental in the *canonical group quantazation method* developed by C.J. Isham.

Recall the obstruction for the lift map $\hat{J} : \mathfrak{g} \longrightarrow C^\infty(M)$ to be a Lie algebra homomorphism is measured by a bilinear map $z : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ which is antisymmetric and satisfies the Jacobi identity. If we consider $\mathbb{R}$ as a trivial $\mathfrak{g}$-module, then it is easily seen that z defines a cohomology class in $[z] \in H^2(\mathfrak{g}, \mathbb{R})$ (Chevalley and Eilenberg cohomology). We showed that this 2-cocycle can be made to vanish if we could find an element $h \in \mathfrak{g}^*$ such that $z(A, B) = h([A, B])$ for all $A, B \in \mathfrak{g}$, which is actually equivalent as requiring that the cohomology class defined by z is the zero class.

If the cohomology class defined by the action of the group G is not the zero class we can make a little trick called a **central extension of the Lie algebra**. The idea is to "enlarge the group in such a way that the Poisson algebra bracket the new group does close"[3]. More precisely, we can consider a central extension of $\mathfrak{g}$ by $\mathbb{R}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & C^\infty(M) \arrow[r, "j"] & \text{Ham VF}(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\alpha"] \arrow[u] & \mathfrak{g} \oplus \mathbb{R} \arrow[r, "\beta"] \arrow[u, "\hat{J}_{\mathbb{R}}"] & \mathfrak{g} \arrow[r] \arrow[u, "\gamma"'] & 0
\end{tikzcd} \tag{3.1}$$

where $\alpha(r) = (0, r)$ and $\beta(A, r) = A$.

We will define a Lie bracket on $\mathfrak{g} \oplus \mathbb{R}$ by

$$[(A, r), (B, s)] = ([A, B], z(A, B)).$$

The new momentum map $\hat{J}_{\mathbb{R}}$, constructed from the old one $\hat{J}$, is defined by

$$\hat{J}_{\mathbb{R}}(A, r) = \hat{J}(A) = r.$$

Finally, if we compute

$$\begin{aligned}
\{\hat{J}_{\mathbb{R}}(A, r), \hat{J}_{\mathbb{R}}(B, s)\} &= \{\hat{J}(A) + r, \hat{J}(B) + s\} = \{\hat{J}(A), \hat{J}(B)\} \\
&= \hat{J}([A, B]) + z(A, B) = \hat{J}_{\mathbb{R}}([A, B], z(A, B)) \\
&= \hat{J}_{\mathbb{R}}([(A, r), (B, s)]),
\end{aligned}$$

we see that the desired property for diagram (3.1) is satisfied.

References

[1] Abraham, R. and Mardsen, J.E., *Foundation of Mechanics*, Addison-Wesley, 1987.

[2] Berndt, R., *Introduction to Symplectic Geometry*, AMS, Graduate Studies in Mathematics, Volume 26, 1987.

[3] Isham, C.J., *Topological and Global Aspects of Quantum Theory*, Relativity, Groups and Tpology II, Elsevier Science Publishers B.V., 1984.

[4] Puta, M., *Hamiltonian Mechanical Systems and Geometric Quantization*, Kluwer Academic Publishers, 1993.

[5] Warner, F., *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New York, 1983.

Berlin Mathematical School, Berlin, Germany.
E-mail address: juanitorduz@gmail.com