S^1-EQUIVARIANT DIRAC OPERATORS ON THE HOPF FIBRATION

JUAN CAMILO ORDUZ

Abstract. In this expository article we discuss the fundamentals of the Hopf fibration with particular emphasis on the Dirac-type operators induded, in the sense of [7], by the Hodge-de Rham and spin-Dirac operator. In addition, we compute the Dirac-Schrödinger type operator introduced in [19] and [20].

Contents

1. The Levi-Civita connection

In this first section we describe the Levi-Civita connection of the standard round metrics of the spheres S^2 and S^3. In particular, we well compute the components of the connection 1-form in an appropriate local orthonormal basis. These expressions will be used afterwards to compute the spin connection for the associated spinor bundles.

1.1. **Round metric on S^2.** Let us consider a 2-sphere of radius $r > 0$

$$S^2(r) := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = r^2\} \subset \mathbb{R}^3,$$

equipped with the induced Riemanian metric from $\mathbb{R}^3$. With respect to a local parametrization given by polar coordinates

$$\begin{aligned} x_1(r,\theta,\phi) &:= \cos\phi\sin\theta, \\ x_2(r,\theta,\phi) &:= \sin\phi\sin\theta, \\ x_3(r,\theta,\phi) &:= \cos\theta, \end{aligned} \tag{1.1}$$

where $0 < \theta < \pi$ and $0 < \phi < 2\pi$, the metric can be written as ([9])

$$g^{TS^2(r)} = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1.2}$$

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The only non-vanishing Christoffel symbols associated to this metric are ([10])

$$\begin{aligned}\Gamma^{\theta}_{\phi\phi} =& \frac{1}{2r^2}(-\partial_\theta(r^2\sin^2\theta)) = -\sin\theta\cos\theta,\\ \Gamma^{\phi}_{\theta\phi} =& \frac{1}{2r^2\sin^2\theta}(\partial_\theta(r^2\sin^2\theta)) = \cot\theta.\end{aligned}$$

Hence, we have the following explicit formulas for the Levi-Civita connection:

$$\begin{aligned}\nabla_{\partial_\theta}\partial_\theta =& 0,\\ \nabla_{\partial_\phi}\partial_\phi =& -\sin\theta\cos\theta\partial_\theta,\\ \nabla_{\partial_\theta}\partial_\phi =& \cot\theta\partial_\phi,\\ \nabla_{\partial_\phi}\partial_\theta =& \cot\theta\partial_\phi.\end{aligned} \tag{1.3}$$

Let us consider the following local orthonotmal basis for $TS^2(r)$

$$\begin{aligned}e_1 :=& \frac{\partial_\theta}{r},\\ e_2 :=& \frac{\partial_\phi}{r\sin\theta},\end{aligned}$$

with associated dual frame

$$\begin{aligned}e^1 :=& rd\theta,\\ e^2 :=& r\sin\theta d\theta.\end{aligned}$$

For further reference we compute the exterior derivative

$$\begin{aligned}de^1 &= 0,\\ de^2 &= d(r\sin\theta d\phi) = r\cos\theta d\theta\wedge d\phi = \frac{\cot\theta}{r}e^1\wedge e^2.\end{aligned}$$

In this orthonormal basis the volume form is $\text{vol}_{S^2(r)} = e^1\wedge e^2$. To be precise, we consider the orientation such that $\text{vol}_{\mathbb{R}^3} = rdr\wedge\text{vol}_{S^2(r)}$.

We want now calculate the components $\omega_{ij}\in\Omega^1(S^2)$ of the connection 1-form associated with this basis. These components defined by the relations

$$\nabla e_j =: \omega_{ij}\otimes e_i, \tag{1.4}$$

where the sum over repeated indices is understood. From (1.3) and (1.4) we get

$$\begin{aligned}\nabla_{e_1}e_1 &= 0,\\ \nabla_{e_1}e_2 &= \partial_\theta\left(\frac{1}{r^2\sin\theta}\right)\partial_\phi + \left(\frac{1}{r^2\sin\theta}\right)\nabla_{\partial_\theta}\partial_\phi = -\frac{\cos\theta}{r^2}\partial_\phi + \frac{\cos\theta}{r^2}\partial_\phi = 0,\\ \nabla_{e_2}e_1 &= \left(\frac{1}{r^2\sin\theta}\right)\nabla_{\partial_\phi}\partial_\theta = \left(\frac{1}{r^2\sin\theta}\right)\cot\theta\partial_\phi = \frac{\cot\theta}{r}e_2,\\ \nabla_{e_2}e_2 &= \left(\frac{1}{r\sin\theta}\right)^2\nabla_{\partial_\phi}\partial_\phi = -\left(\frac{1}{r\sin\theta}\right)^2\sin\theta\cos\theta\partial_\theta = -\frac{\cot\theta}{r}e_1.\end{aligned}$$

We can read from the expressions above

$$\omega_{12} = -\omega_{21} = -\frac{\cot\theta}{r}e^2. \tag{1.5}$$

Remark 1.1 (Structure Equations)**.** Since the Levi-Civita connection is compatible with metric, then $\omega_{ij} = -\omega_{ji}$. On the other hand, since it is also torsion-free one can verify that the components ω_{ij} satisfy the structure equation ([17, Proposition 5.32])

$$de^i + \omega_{ij} \wedge e^j = 0. \tag{1.6}$$

Note for example for $i = 2$,

$$de^2 + \omega_{21} \wedge e^1 = \frac{\cot\theta}{r}e^1 \wedge e^2 + \frac{\cot\theta}{r}e^2 \wedge e^1 = 0.$$

From the components of the connection 1-form we can compute the components $\Omega_{ij} \in \Omega^2(S^2)$ of the curvature using the relation ([17, Proposition 5.21])

$$\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}. \tag{1.7}$$

In this particular case, the only non-zero component is

$$\Omega_{12} = d\omega_{12} = \frac{\csc^2\theta}{r^2}e^1 \wedge e^2 - \frac{\cot\theta}{r}de^2 = \frac{\csc^2\theta}{r^2}e^1 \wedge e^2 - \frac{\cot^2\theta}{r^2}e^1 \wedge e^2 = \frac{1}{r^2}e^1 \wedge e^2.$$

Remark 1.2 (Gauß-Bonnet Theorem)**.** If we integrate the 2-form $\Omega_{12}/2\pi$ over $S^2(r)$ we obtain

$$\int_{S^2(r)} \frac{\Omega_{12}}{2\pi} = \frac{1}{2\pi r^2}\int_{S^2(r)} e^1 \wedge e^2 = \frac{4\pi r^2}{2\pi r^2} = 2,$$

which verifies the Gauß-Bonnet theorem since the Euler characteristic as $\chi(S^2(r)) = 2$ for any $r > 0$.

1.2. **Round metric on S^3.** We now consider the 3-sphere

$$S^3 := \{(z_0, z_1) \ : \ |z_0|^2 + |z_1|^2 = 1\} \subset \mathbb{C}^2$$

with the induced round metric. We want to proceed as above in order to calculate explicitly the connection 1-form and the curvature. We introduce a local parametrization, the so-called **Hopf coordinates**, of S^3 given by

$$\begin{aligned} z_0(\xi_1, \xi_2, \eta) &:= e^{i\xi_1}\cos\eta, \\ z_1(\xi_1, \xi_2, \eta) &:= e^{i\xi_2}\sin\eta, \end{aligned} \tag{1.8}$$

where $0 < \xi_1, \xi_2 < 2\pi$ and $0 < \eta < \pi/2$. The metric induced from $\mathbb{C}^2 \cong \mathbb{R}^4$ can be written as

$$g^{TS^3} = \cos^2\eta d\xi_1^2 + \sin^2\eta d\xi_2^2 + d\eta^2. \tag{1.9}$$

1.2.1. *The volume form.* We now want to determine the induced orientation on S^3 from the standard orientation of $\mathbb{C}^2$ with respect to the coordinates (1.8). To do this we it is enough to compute the volume form. Is it easy to see from (1.9) that $\text{vol}_{S^3} = \pm \sin\eta\cos\eta d\xi_1 \wedge d\xi_2 \wedge d\eta$, but we want to pick the sign such that $\text{vol}_{\mathbb{C}^2} = r^3 dr \wedge \text{vol}_{S^3}$ where r denotes the radial coordinate. If we define

$$
\begin{aligned}
z_0(r) &\coloneqq r z_0 = r e^{i\xi_1}\cos\eta,\\
z_1(r) &\coloneqq r z_1 = r e^{i\xi_2}\sin\eta,
\end{aligned}
$$

then we have [11, Section V.1]

$$
\text{vol}_{\mathbb{C}^2} = \left(\frac{i}{2}\right)^2 dz_0(r)\wedge d\bar{z}_0(r)\wedge dz_1(r)\wedge d\bar{z}_1(r). \tag{1.10}
$$

Therefore, we need to compute the form $dz_0(r)\wedge d\bar{z}_0(r)\wedge dz_1(r)\wedge d\bar{z}_1(r)$. First, note that

$$
\begin{aligned}
dz_0(r) &= z_0 dr + r dz_0,\\
d\bar{z}_0(r) &= \bar{z}_0 dr + r d\bar{z}_0,\\
dz_1(r) &= z_1 dr + r dz_1,\\
d\bar{z}_1(r) &= \bar{z}_1 dr + r d\bar{z}_1.
\end{aligned}
$$

On the other hand we can express

$$
\begin{aligned}
dz_0 =& i z_0 d\xi_1 - \tan\eta z_0 d\eta,\\
d\bar{z}_0 =& - i\bar{z}_0 d\xi_1 - \tan\eta \bar{z}_0 d\eta,\\
dz_1 =& i z_1 d\xi_2 + \cot\eta z_1 d\eta,\\
d\bar{z}_1 =& - i\bar{z}_1 d\xi_2 + \cot\eta \bar{z}_1 d\eta.
\end{aligned}
$$

Using the relations $|z_0|^2 = \cos^2\eta$ and $|z_1|^2 = \sin^2\eta$ we calculate

$$
\begin{aligned}
dz_0(r)\wedge d\bar{z}_0(r) =& r z_0 dr\wedge d\bar{z}_0 - r\bar{z}_0 dr\wedge dz_0 + r^2 dz_0\wedge d\bar{z}_0\\
=& r dr\wedge(-i\cos^2\eta d\xi_1 - \sin\eta\cos\eta d\eta)\\
&- r dr\wedge(i\cos^2\eta d\xi_1 - \sin\eta\cos\eta d\eta)\\
&+ r^2(-2i\sin\eta\cos\eta d\xi_1\wedge d\eta)\\
=& -2ir\cos^2\eta dr\wedge d\xi_1 - 2ir^2\sin\eta\cos\eta d\xi_1\wedge d\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
dz_1(r)\wedge d\bar{z}_1(r) =& r z_1 dr\wedge d\bar{z}_1 - r\bar{z}_1 dr\wedge dz_1 + r^2 dz_1\wedge d\bar{z}_1\\
=& r dr\wedge(-i\sin^2\eta d\xi_2 + \sin\eta\cos\eta d\eta)\\
&+ r dr\wedge(i\sin^2\eta d\xi_2 + \sin\eta\cos\eta d\eta)\\
&r^2(2i\sin\eta\cos\eta d\xi_2\wedge d\eta)\\
=& -2ir\sin^2\eta dr\wedge d\xi_2 + 2ir^2\sin\eta\cos\eta d\xi_2\wedge d\eta.
\end{aligned}
$$

Hence, we find

$$\begin{aligned}dz_0(r)\wedge d\bar{z}_0(r)\wedge dz_1(r)\wedge d\bar{z}_1(r) &= 4r^3\sin\eta\cos^3\eta dr\wedge d\xi_1\wedge d\xi_2\wedge d\eta\\ &\quad - 4r^3\sin^3\eta\cos\eta d\xi_1\wedge d\eta\wedge dr\wedge d\xi_2\\ &= 4r^3\sin\eta\cos\eta dr\wedge d\xi_1\wedge d\xi_2\wedge d\eta.\end{aligned}$$

We conclude from (1.10) that the desired volume form is

$$\mathrm{vol}_{S^3} = -\sin\eta\cos\eta d\xi_1\wedge d\xi_2\wedge d\eta. \tag{1.11}$$

In particular,

$$\begin{aligned}\mathrm{vol}(S^3) =& \int_{S^3}(-\sin\eta\cos\eta d\xi_1\wedge d\xi_2\wedge d\eta)\\ =& \int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}\sin 2\eta d\xi_1 d\xi_2 d\eta\\ =&(2\pi)^2\left(-\frac{1}{4}\cos 2\eta\Big|_0^{\pi/2}\right)\\ =&2\pi^2.\end{aligned}$$

1.2.2. *The connection* 1 *-form.* Our next task is to compute the connection 1-form of the Levi-Civita connection associated to the metric (1.9) with respect to a convenient orthonormal basis, which will become natural in study the Hopf fibration below. To begin, with we compute the non-vanishing Christoffel symbols

$$\begin{aligned}\Gamma^{\eta}_{\xi_1\xi_1} =&\frac{1}{2}(-\partial_\eta(\cos^2\eta)) = \sin\eta\cos\eta,\\ \Gamma^{\eta}_{\xi_2\xi_2} =&\frac{1}{2}(-\partial_\eta(\sin^2\eta)) = -\sin\eta\cos\eta,\\ \Gamma^{\xi_1}_{\xi_1\eta} =&\frac{1}{2\cos^2\eta}(\partial_\eta(\cos^2\eta)) = -\tan\eta,\\ \Gamma^{\xi_2}_{\xi_2\eta} =&\frac{1}{2\sin^2\eta}(\partial_\eta(\sin^2\eta)) = \cot\eta.\end{aligned}$$

As before, the relations above imply the explicit action of the Levi-Civita connection on the induced coordinate vector fields

$$\begin{aligned}\nabla_{\partial_{\xi_1}}\partial_{\xi_1} &= \sin\eta\cos\eta\partial_\eta,\\ \nabla_{\partial_{\xi_2}}\partial_{\xi_2} &= -\sin\eta\cos\eta\partial_\eta,\\ \nabla_{\partial_\eta}\partial_{\xi_1} &= -\tan\eta\partial_{\xi_1},\\ \nabla_{\partial_\eta}\partial_{\xi_2} &= \cot\eta\partial_{\xi_2}.\end{aligned} \tag{1.12}$$

Consider the local orthonormal basis

$$\begin{aligned}e_1 &:= \partial_\eta,\\ e_2 &:= \tan\eta\partial_{\xi_1} - \cot\eta\partial_{\xi_2},\\ e_3 &:= \partial_{\xi_1} + \partial_{\xi_2}.\end{aligned} \tag{1.13}$$

Remark 1.3. To verify the orthonormality condition observe for example

$$\langle e_2, e_3\rangle = \tan\eta\langle\partial_{\xi_1}, \partial_{\xi_1}\rangle - \cot\eta\langle\partial_{\xi_2}, \partial_{\xi_2}\rangle = \tan\eta\cos^2\eta - \cot\eta\sin^2\eta = 0.$$

The associated dual basis of (1.13) is

$$\begin{aligned} e^1 :=& d\eta, \\ e^2 :=& \frac{1}{2}\sin 2\eta(d\xi_1 - d\xi_2), \\ e^3 :=& \cos^2\eta d\xi_1 + \sin^2\eta d\xi_2. \end{aligned}$$

The corresponding exterior derivatives are

$$\begin{aligned} de^1 =& 0, \\ de^2 =& \cos 2\eta d\eta\wedge(d\xi_1 - d\xi_2) = 2\cot 2\eta e^1\wedge e^2, \\ de^3 =& -2\sin\eta\cos\eta d\eta\wedge(d\xi_1 - d\xi_2) = -2e^1\wedge e^2. \end{aligned} \tag{1.14}$$

Remark 1.4 (Volume form)**.** Note in particular

$$\begin{aligned} e^1\wedge e^2\wedge e^3 =& d\eta\wedge(\sin\eta\cos\eta(d\xi_1 - d\xi_2))\wedge(\cos^2\eta d\xi_1 + \sin^2 2\eta d\xi_2) \\ =& \sin^3\eta\cos\eta d\eta\wedge d\xi_1\wedge d\xi_2 - \sin\eta\cos^3\eta d\eta\wedge d\xi_2\wedge d\xi_1 \\ =& \sin\eta\cos\eta d\eta\wedge d\xi_1\wedge d\xi_2. \end{aligned}$$

so we see that $\text{vol}_{S^3} := -e^1\wedge e^2\wedge e^3$.

Remark 1.5. The following trigonometric relations will be needed later:

$$\begin{aligned} \tan\eta - \cot\eta =& \frac{\sin^2\eta - \cos^2\eta}{\sin\eta\cos\eta} = -2\cot 2\eta. \\ \tan\eta + \cot\eta =& \frac{\sin^2\eta + \cos^2\eta}{\sin\eta\cos\eta} = 2\csc 2\eta. \\ \tan^2\eta - \cot^2\eta =& (\tan\eta - \cot\eta)(\tan\eta + \cot\eta) = -4\csc 2\eta\cot 2\eta. \end{aligned}$$

Now we use (1.12) to compute the components $\omega_{ij}\in\Omega^1(S^3)$ of the connection 1-form with respect to this basis (see (1.4)). For ∇e_1 we get

$$\begin{aligned} \nabla_{e_1}e_1 =& 0, \\ \nabla_{e_2}e_1 =& \tan\eta\nabla_{\partial_{\xi_1}}\partial_\eta - \cot\eta\nabla_{\partial_{\xi_2}}\partial_\eta = -(\tan^2\eta\partial_{\xi_1} + \cot^2\eta\partial_{\xi_2}) \\ =& -(\tan\eta - \cot\eta)(\tan\eta\partial_{\xi_1} - \cot\eta\partial_{\xi_2}) - (\partial_{\xi_1} + \partial_{\xi_2}) = 2\cot 2\eta e_2 - e_3. \\ \nabla_{e_3}e_1 =& \nabla_{\partial_{\xi_1}}\partial_\eta + \nabla_{\partial_{\xi_2}}\partial_\eta = -\tan\eta\partial_{\xi_1} + \cot\eta\partial_{\xi_2} = -e_2. \end{aligned}$$

For ∇e_2 we compute similarly

$$\begin{aligned} \nabla_{e_1}e_2 =& \sec^2\eta\partial_{\xi_1} + \tan\eta\nabla_{\partial_\eta}\partial_{\xi_1} + \csc^2\eta\partial_{\xi_2} - \cot\eta\nabla_{\partial_\eta}\partial_{\xi_2} \\ =& (\sec^2\eta - \tan^2\eta)\partial_{\xi_1} + (\csc^2\eta - \cot^2\eta)\partial_{\xi_2} = \partial_{\xi_1} + \partial_{\xi_2} = e_3, \\ \nabla_{e_2}e_2 =& \tan^2\eta\nabla_{\partial_{\xi_1}}\partial_{\xi_1} + \cot^2\eta\nabla_{\partial_{\xi_2}}\partial_{\xi_2} = \sin\eta\cos\eta(\tan^2\eta - \cot^2\eta)\partial_\eta = -2\cot 2\eta e_1, \\ \nabla_{e_3}e_2 =& \tan\eta\nabla_{\partial_{\xi_1}}\partial_{\xi_1} - \cot\eta\nabla_{\partial_{\xi_2}}\partial_{\xi_2} = \sin\eta\cos\eta(\tan\eta + \cot\eta)e_1 = e_1. \end{aligned}$$

and finally for ∇e_3

$$\begin{aligned}
\nabla_{e_1} e_3 =&\nabla_{\partial_\eta}\partial_{\xi_1} + \nabla_{\partial_\eta}\partial_{\xi_2} = -\tan\eta\partial_{\xi_1} + \cot\eta\partial_{\xi_2} = -e_2,\\
\nabla_{e_2} e_3 =& \tan\eta\nabla_{\partial_{\xi_1}}\partial_{\xi_1} - \cot\eta\nabla_{\partial_{\xi_2}}\partial_{\xi_2} = \sin\eta\cos\eta(\tan\eta+\cot\eta)\partial_\eta = e_1,\\
\nabla_{e_3} e_3 =&0.
\end{aligned}$$

From these expressions we can read the components:

$$\begin{aligned}
\omega_{12} =& -2\cot 2\eta e^2 + e^3,\\
\omega_{13} =&e^2,\\
\omega_{23} =& -e^1.
\end{aligned} \tag{1.15}$$

Remark 1.6. Using (1.14), let us verify the structure equations (1.6),

$$\begin{aligned}
de^1 =& -\omega_{12}\wedge e^2 - \omega_{13}\wedge e^3 = (2\cot 2\eta e^2 - e^3)\wedge e^2 - e^2\wedge e^3 = 0,\\
de^2 =& -\omega_{21}\wedge e^1 - \omega_{23}\wedge e^3 = (-2\cot 2\eta e^2 + e^3)\wedge e^1 + e^1\wedge e^3 = 2\cot 2\eta e^1\wedge e^2,\\
de^3 =& -\omega_{31}\wedge e^1 - \omega_{32}\wedge e^2 = e^2\wedge e^1 - e^1\wedge e^2 = -2e^1\wedge e^2.
\end{aligned}$$

1.2.3. *The curvature.* To end this section, we calculate the components $\Omega_{ij}\in\Omega^2(S^3)$ of the curvature using (1.7). First we compute the exterior derivative of ω_{ij},

$$\begin{aligned}
d\omega_{12} &= 4\csc^2 2\eta e^1\wedge e^2 - 2\cot 2\eta e^2\wedge de^2 + de^3 = 2(2\csc^2 2\eta - 1)e^1\wedge e^2,\\
d\omega_{13} &= de^2 = 2\cot 2\eta e^1\wedge e^2,\\
d\omega_{23} &= d(-e^1) = 0.
\end{aligned}$$

Finally, a straight forward computations show that

$$\begin{aligned}
\Omega_{12} =&d\omega_{12} + \omega_{13}\wedge\omega_{32} = 2(2\csc^2 2\eta - 1)e^1\wedge e^2 + e^2\wedge e^1 = (4\csc^2 2\eta - 3)e^1\wedge e^2,\\
\Omega_{13} =&d\omega_{13} + \omega_{12}\wedge\omega_{23} = (-2\cot 2\eta e^2 + e^3)\wedge(-e^1) = -2\cot 2\eta e^1\wedge e^2 + e^1\wedge e^3,\\
\Omega_{23} =&d\omega_{23} + \omega_{21}\wedge\omega_{13} = 2\cot 2\eta e^1\wedge e^2 + (2\cot 2\eta e^2 - e^3)\wedge e^2 = 2\cot 2\eta e^1\wedge e^2 + e^2\wedge e^3.
\end{aligned}$$

2. Topology and geometry of the Hopf fibration

In this section we introduce the Hopf fibration and establish notation. The Hopf fibration is a well-known case of study where many of geometric and topological quantities can be explicitly worked calculated. For more complete treatments refer for example to [6, Section III.17] and [18, Section 9.4]. Concretely, the Hopf fibration

$$\begin{tikzcd}
S^1 \arrow[r, hook] & S^3 \arrow[d, "\pi"]\\
& S^2
\end{tikzcd}$$

is a non-trivial S^1-principal bundle over S^2. We will verify this by computing the transition functions explicitly. In addition, if we endow S^3 with the round metric (1.9) introduced in Section 1.2, we will see that the metric that we need to consider on S^2 so that $\pi : S^3 \longrightarrow S^2$ becomes a Riemannian submersion is precisely (1.2) with $r = 1/2$. Next we will use this result to compute the corresponding Chern class using Chern-Weil theory ([17, Chapter 6]). Finally we will compute the Hopf invariant associated to π.

2.1. **Definition and properties of the Hopf fibration.** Let us consider an action of $S^1 \coloneqq \{\lambda \in \mathbb{C} \,:\, |\lambda| = 1\} \subset \mathbb{C}$ on S^3 defined for $\lambda \in S^1$ and $(z_0, z_1) \in S^3$ by

$$\lambda(z_0, z_1) \coloneqq (\lambda z_0, \lambda z_1).$$

It is clear that this action is orientation and metric preserving. In addition, this action is free and the quotient space S^3/S^1 can be equipped with a unique smooth structure such that the orbit map $\pi : S^3 \longrightarrow S^3/S^1$ is smooth ([25, Theorem 3.58]). Indeed, one can identify $S^3/S^1 \cong S^2$.

Proposition 2.1. *The orbit map is given explicitly by*

$$\begin{aligned} \pi : \quad & S^3 \longrightarrow S^2 \\ & (z_0, z_1) \longmapsto (a, b) \coloneqq (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2). \end{aligned} \tag{2.1}$$

This function is known as the **Hopf map** *(*[6, Section III.17]*).*

Proof. First of all note that, since $|z_0|^2 + |z_1|^2 = 1$, then

$$|a|^2 + |b|^2 = 4|z_0|^2|z_1|^2 + (|z_0|^2 - |z_1|^2)^2 = (|z_0|^2 + |z_1|^2)^2 = 1,$$

which shows that so the map π is well-defined. Now let us see why π as defined above is the orbit map. For $\lambda \in S^1 \subset \mathbb{C}$ we trivially have $\pi(\lambda z_0, \lambda z_1) = \pi(z_0, z_1)$. On the other hand, let us assume that $\pi(z_0, z_1) = \pi(w_0, w_1)$. We want to show that there exists $\lambda \in S^1$ such that $(z_0, z_1) = \lambda(w_0, w_1)$. Let us write these points in polar coordinates as $z_k = r_k e^{i\varphi_k}$ and $w_k = s_k e^{i\psi_k}$ with $r_k, s_k \geq 0$ for $k = 0, 1$. One easily verifies that the following conditions must hold true

$$\begin{aligned} e^{i(\varphi_0 - \varphi_1)} r_0 r_1 &= e^{i(\psi_0 - \psi_1)} s_0 s_1, \\ r_0^2 - r_1^2 &= s_0^2 - s_1^2, \\ r_0^2 + r_1^2 &= s_0^2 + s_1^2 = 1. \end{aligned}$$

The last two equations imply $r_0 = s_0$ and $r_1 = s_1$. Finally, note we can write

$$z_k = r_k e^{i\varphi_k} = s_k e^{i\varphi_k} = (e^{i(\phi_k - \psi_k)}) s_k e^{i\psi_k},$$

so the claim follows taking $\lambda \coloneqq e^{i(\phi_k - \psi_k)}$, which y the firs equation above we know is independent of k. □

With respect to the coordinates (1.8) we can write the Hopf map as

$$\pi(e^{i\xi_1}\cos\eta, e^{i\xi_2}\sin\eta) = (e^{i(\xi_1 - \xi_2)}\sin(2\eta), \cos(2\eta)), \tag{2.2}$$

from where we see, in view of (1.1), that the Hopf map π defined above is surjective.

2.2. **S^1-bundle structure.** In this subsection we show that the map $\pi : S^3 \longrightarrow S^2$ is a non-trivial S^1-principal bundle. To do this we will compute the transition functions following [18, Example 9.9]. Consider two local charts on S^2 using the spherical coordinates (1.1), but with different choices domains for the parameters θ and ϕ. For $\epsilon > 0$ small enough define

$$\begin{aligned} U_N &:= \{x(\theta,\phi) \mid 0 \leq \theta \leq \pi/2 + \epsilon,\ 0 \leq \phi < 2\pi\}, \\ U_S &:= \{x(\theta,\phi) \mid \pi/2 - \epsilon \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi\}. \end{aligned} \tag{2.3}$$

The sets U_N and U_S are the northern and southern hemispheres respectively and the intersection $U_N \cap U_S$ is an equator strip which is homeomorphic to S^1. Note that if $(z_0, z_1) \in \pi^{-1}(U_N)$, it follows from (2.2) that $z_0 \neq 0$. Analogously, if $(z_0, z_1) \in \pi^{-1}(U_S)$ then $z_1 \neq 0$. We can therefore define two local trivializations

$$\begin{aligned} \varphi_N :\ \pi^{-1}(U_N) &\longrightarrow U_N \times S^1 \\ (z_0, z_1) &\longmapsto \left((\pi(z_0,z_1), \frac{z_0}{|z_0|} \right) \end{aligned}$$

and

$$\begin{aligned} \varphi_S :\ \pi^{-1}(U_S) &\longrightarrow U_S \times S^1 \\ (z_0, z_1) &\longmapsto \left((\pi(z_0,z_1), \frac{z_1}{|z_1|} \right). \end{aligned}$$

If $(z_0, z_1) \in \pi^{-1}(U_N \cap U_Z)$ then $|z_0| = |z_1| = \sqrt{2}/2$ since this correspond to the value $\eta = \pi/4$. Thus

$$\varphi_N \circ \varphi_S^{-1} \left((\pi(z_0,z_1), \sqrt{2} z_1 \right) = \left((\pi(z_0,z_1), \sqrt{2} z_0 \right).$$

This shows that the transition function $g_{NS} : U_N \cap U_S \longrightarrow S^1$ defined by the relation

$$\varphi_S^{-1}(x,\lambda) = \varphi_N^{-1}(x, g_{NS}(x)\lambda) \quad \text{for } (x,\lambda) \in U_N \cap U_S \times S^1,$$

is given by

$$g_{NS}(\pi(z_0,z_1)) = \frac{z_0}{z_1} \in S^1.$$

Since the transition function which generates a trivial bundle $S^2 \times S^1$ is $g_{NS} = 1 \in S^1 \subset \mathbb{C}$ we see that the Hopf map is not trivial.

2.3. **Induced Riemannian metric.** Let us consider now S^3 equipped with the metric studied in Section 1.2. We now describe a metric on the quotient space S^2 so that the Hopf map becomes a Riemannian fibration, i.e. such that $d\pi : \ker(d\pi)^{\perp} \longrightarrow TS^2$ is an isometry. In view of the expression of the Hopf map in the coordinates (1.8), $\pi(e^{i\xi_1}\cos\eta, e^{i\xi_2}\sin\eta) = (e^{i(\xi_1-\xi_2)}\sin(2\eta), \cos(2\eta))$, we define functions $\theta = \theta(\xi_1,\xi_2,\eta)$ and $\phi = \phi(\xi_1,\xi_2,\eta)$ to parametrize the image of π as

$$(\theta,\phi) \longmapsto (e^{i\phi(\xi_1,\xi_2,\eta)}\sin(\theta(\xi_1,\xi_2,\eta)), \cos(\theta(\xi_1,\xi_2,\eta))),$$

son that in the (θ,ϕ)-coordinates,

$$\pi(\xi_1,\xi_2,\eta)=(\theta(\xi_1,\xi_2,\eta),\phi(\xi_1,\xi_2,\eta))=(2\eta,\xi_1-\xi_2).$$

With respect to the the coordinate local basis $\{\partial_{\xi_1},\partial_{\xi_2},\partial_\eta\}$ and $\{\partial_\theta,\partial_\phi\}$ the derivative of the Hopf map π is

$$d\pi(\xi_1,\xi_2,\eta)=\begin{pmatrix} 0 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

In particular,

$$\begin{aligned} d\pi(\xi_1,\xi_2,\eta)(\partial_{\xi_1}+\partial_{\xi_2})&=0,\\ d\pi(\xi_1,\xi_2,\eta)(\partial_{\xi_1}-\partial_{\xi_2})&=2\partial_\phi,\\ d\pi(\xi_1,\xi_2,\eta)\partial_\eta&=2\partial_\theta. \end{aligned}$$

The first of these relations is not surprising since $X=-(\partial_{\xi_1}+\partial_{\xi_2})$ is the **generating vector field** of the action and therefore is tangent to the fibers. More precisely, the generator vector field X of the S^1-action is defined to act on a on a smooth function $f\in C^\infty(S^3)$ as ([4, Equation (1.2)])

$$\begin{aligned} (Xf)(\xi_1,\xi_2,\eta) :=&\frac{d}{dt}f(e^{-it}(\xi_1,\xi_2,\eta))\bigg|_{t=0}\\ =&\frac{d}{dt}f(\xi_1-t,\xi_2-t,\eta)\bigg|_{t=0}\\ =&-\partial_{\xi_1}f(\xi_1,\xi_2,\eta)-\partial_{\xi_2}f(\xi_1,\xi_2,\eta). \end{aligned}$$

Remark 2.1 (Mean curvature form)**.** Since $\|X\|=1$, the **mean curvature** 1-form $\kappa:=-d\log(\|X\|)$ vanishes, which shows that the S^1-fibers are totally geodesic.

The requirement on the metric $\langle\cdot,\cdot\rangle_{S^2}$ to ensure π to be a Riemannian fibration is $\langle Y,Y\rangle_{S^3}=\langle d\pi(Y),d\pi(Y)\rangle_{S^2}$ for all vector fields Y orthogonal to $\partial_{\xi_1}+\partial_{\xi_2}$, i.e. **horizontal vector fields**. Let us consider the local orthonormal basis (1.13). Observe that $e_1=\partial_\eta$ and $e_2=\tan\eta\partial_{\xi_1}-\cot\eta\partial_{\xi_2}$ are horizontal unit vector fields. Hence, their image under $d\pi$ must also have norm one. Since

$$\begin{aligned} d\pi(e_1)=&(\tan\eta+\cot\eta)\partial_\phi=2\csc 2\eta\partial_\phi,\\ d\pi(e_2)=&2\partial_\theta, \end{aligned}$$

the conditions require are

$$\begin{aligned} \|2\csc 2\eta\partial_\phi\|_{S^2}&=1,\\ \|2\partial_\theta\|_{S^2}&=1. \end{aligned}$$

That is, we need to equip S^2 with the Riemannian metric

$$g^{TS^2(1/2)}=\frac{1}{4}d\theta^2+\frac{1}{4}\sin^2\theta d\phi^2. \tag{2.4}$$

In summary, the image of the Hopf map is a 2-sphere of radius 1/2 with the usual round metric of Section 1.1.

2.4. **Connection form and curvature.**

2.4.1. *The Levi-Civita connection.* The metric on S^3 defines a connection on the principal bundle $\pi : S^3 \longrightarrow S^2$ ([3, Beispiel 3.3]). We want to find the associated connection 1-form and calculate the curvature. First, observe that the Lie algebra $\mathfrak{s}^1$ of S^1 is can be identified $i\mathbb{R}$ (exponential map). The **horizontal space** H defined by the metric on S^3 is the bundle generated by the horizontal vector fields e_1 and e_2. We aim to find the connection 1-form $\omega \in \Omega^1(S^3, \mathfrak{s}^1)$ associated to this horizontal bundle defined by the requirements $\ker \omega = H$ and $\omega(X) = i \in \mathfrak{s}^1$. The natural choice is $\omega := -ie^3$. For example,

$$-ie^3(X) = i(\cos^2 \eta d\xi_1 + \sin^2 \eta d\xi_2)(-\partial_{\xi_1} - \partial_{\xi_2}) = i(\cos^2 \eta + \sin^2 \eta) = i \in \mathfrak{s}^1.$$

The curvature of this connection is $\Omega := d\omega = 2ie^1 \wedge e^2 \in \Omega^2(S^3, \mathfrak{s}^1)$. Note that since

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = -2 \cot 2\eta e_2 + 2e_3,$$

then the projection P onto the vertical space is

$$P([e_1, e_2]) = 2e_3 = -2X,$$

which shows that the curvature satisfies

$$\Omega(e_1, e_2) \otimes X = -P([e_1, e_2]) \otimes i = 2iX,$$

as expected from the geometric definition [4, Equation (1.6)].

2.4.2. *The first Chern class.* The curvature form Ω is a **basic** 2-form (see below) so there exist a 2-form $\bar{\Omega} \in \Omega^2(S^2)$ such that $\Omega = \pi^*(\bar{\Omega})$. We want to find $\bar{\Omega}$. First we compute

$$\begin{aligned}\pi^* d\theta &= d\pi^*\theta = d(2\eta) = 2d\eta,\\ \pi^* d\phi &= d\pi^*\phi = d(\xi_1 - \xi_2) = d\xi_1 - d\xi_2.\end{aligned}$$

These expressions imply

$$\pi^*\left(\frac{1}{4}\sin\theta d\theta \wedge d\phi\right) = \frac{1}{4}\sin 2\eta(2d\eta) \wedge (d\xi_1 - d\xi_2) = e^1 \wedge e^2$$

Hence, the 2-form

$$\bar{\Omega} := \frac{i}{2}\sin\theta d\theta \wedge d\phi,$$

satisfies $\Omega = \pi^*(\bar{\Omega})$. We can use this 2-form to construct a representative of the first Chern class of this principal bundle ([17, Definition 3.35])

$$c_1(\pi) := \left[-\frac{\bar{\Omega}}{2\pi i}\right] \in H^2(S^2; \mathbb{Z}).$$

The integral of the first Chern class is ([3, Beispiel 3.10])

$$\int_{S^2}\left(-\frac{\bar{\Omega}}{2\pi i}\right) = -\int_{S^2}\frac{\sin\theta d\theta \wedge d\phi}{4\pi} = -1.$$

In particular, this argument also shows that the Hopf map defines a non-trivial principal bundle.

2.5. **The Hopf invariant.** In this section we compute the so-called **Hopf invariant** of the Hopf map π. We first recall the definition of the Hopf invariant $H(f)$ associated to any map $f : S^3 \longrightarrow S^1$ ([17, Section 5.6 (a)]). First choose a 2-form $\bar{\beta} \in \Omega^2(S^2)$ such that

$$\int_{S^2} \bar{\beta} = 1.$$

Since $df^*(\bar{\beta}) = 0$ and $H^2(S^3; \mathbb{R}) = 0$ then there exists $\alpha \in \Omega^1(S^3)$ such that $f^*(\bar{\beta}) = d\alpha$. Define the Hopf invariant $H(f)$ of f by the formula

$$H(f) := \int_{S^3} \alpha \wedge d\alpha \in \mathbb{R}.$$

The following are two remarkable facts about $H(f)$,

- It does not depend on the choice of $\bar{\beta}$ or α, it just depends on f itself.
- Its value depends only on the homotopy class of f.

Let us now compute $H(\pi)$. We have seen that

$$\int_{S^2} \frac{\sin\theta d\theta \wedge d\phi}{4\pi} = 1.$$

From what we discussed in the last section we have

$$\pi^* \left(\frac{\sin\theta d\theta \wedge d\phi}{4\pi} \right) = \frac{e^1 \wedge e^2}{\pi}.$$

On the other hand by (1.14) we have

$$-\frac{1}{2\pi} de^3 = \frac{e^1 \wedge e^2}{\pi}.$$

Thus, using (1.11), we compute

$$H(\pi) = \int_{S^3} \left(-\frac{e^3}{2\pi} \right) \wedge \left(\frac{e^1 \wedge e^2}{\pi} \right) = \frac{1}{2\pi^2} \int_{S^3} (-e^1 \wedge e^2 \wedge e^3) = \frac{1}{2\pi^2} \int_{S^3} \mathrm{vol}_{S^3} = 1.$$

3. Hodge-de Rham operator

In this section we study the Hodge-de Rham operator, the associated Dirac operator of the usual (left) Clifford structure on the exterior algebra. We implement the procedure described in [7] to "push-down" the operator from S^3 to S^2 thorough π, obtaining an operator whose first order part is simply the Hodge-de Rham operator on S^2 and the zero order term is an edomorphism which depends on the curvature of the Hopf bundle with respect to the connection induced by the metric. At the end of this section we modify this procedure to construct a transversally elliptic operator on S^3 such that the first order part of induced operator is again the Hodge-de Rham operator and with the additional property that the zero order term anti-commutes with the associated chirality operator ([19]).

3.1. **The Clifford module $\wedge T^*S^3 \otimes \mathbb{C}$.** Let us consider the complex vector bundle $\wedge T^*S^3 \otimes \mathbb{C}$ with the Hermitian metric induced from (1.9). For example,

$$\begin{aligned}\langle d\xi_1, d\xi_1\rangle =& \cos^{-2}\eta,\\ \langle d\xi_2, d\xi_2\rangle =& \sin^{-2}\eta,\\ \langle d\eta, d\eta\rangle =& 1,\\ \langle d\xi_1 \wedge d\xi_2, d\xi_1 \wedge d\xi_2\rangle =& \sin^{-2}\eta\cos^{-2}\eta.\end{aligned}$$

Associated with this metric and the orientation (1.11) we consider the **Hodge star operator**

$$* : \wedge^k T^*S^3 \otimes \mathbb{C} \longrightarrow \wedge^{3-k}T^*S^3 \otimes \mathbb{C},$$

defined by the relation

$$\alpha \wedge \overline{*\beta} = \langle \alpha, \beta\rangle \mathrm{vol}_{S^3}.$$

Let $L^2(S^3, \wedge T^*S^3)$ be the Hilbert space completion of $\Omega^*(S^3)$ with respect to the L^2-inner product

$$(\alpha, \beta)_{L^2(S^3,\wedge T^*S^3)} \coloneqq \int_{S^3} \langle \alpha, \beta\rangle \mathrm{vol}_{S^3}. \tag{3.1}$$

The vector bundle $\wedge T^*S^3 \otimes \mathbb{C}$ carries a left Clifford bundle structure with left **Clifford multiplication** $c(\alpha) \coloneqq \alpha \wedge -\iota_{\alpha^\sharp}$ for $\alpha \in \Omega^1(S^3)$ ([4, Section 3.6],[15]). The Clifford multiplication satisfies

$$\begin{aligned}c(\alpha)c(\beta) + c(\beta)c(\alpha) =& -2\langle \alpha, \beta\rangle, \quad \forall \alpha, \beta \in \Omega^1(S^3),\\ c(\alpha)^\dagger =& -c(\alpha) \quad \text{with respect to the } L^2\text{-inner product},\\ c(\alpha)\varepsilon =& -\varepsilon c(\alpha),\end{aligned}$$

where $\varepsilon = (-1)^k$ on k-forms is the **Gauß-Bonnet grading**. To each Clifford bundle we can associate its corresponding **chirality operator** ([4, Lemma 3.17]). In this particular Clifford bundle the chirality operator $\star : \wedge^k_{\mathbb{C}} T^*S^3 \longrightarrow \wedge^{m-k}_{\mathbb{C}} T^*S^3$ is given by ([4, Proposition 3.58])

$$\star \coloneqq (-1)^{3k+k(k-1)/2} i^2 * = -i^{k(k+1)} * \quad \text{on } k\text{-forms}. \tag{3.2}$$

The chirality operator $\star$ satisfies

$$\begin{aligned}\star^2 =& 1,\\ \star^\dagger =& \star,\\ \star(\alpha\wedge)\star =& -\iota_{\alpha^\sharp} \quad \text{for } \alpha \in T^*S^3.\end{aligned} \tag{3.3}$$

Remark 3.1. The right Clifford multiplication, defined by $\widehat{c}(\alpha) \coloneqq \alpha \wedge + \iota_{\alpha^\sharp}$, satisfies:

$$\begin{aligned}\star c(\alpha) &= c(\alpha)\star,\\ \star\widehat{c}(\alpha) &= -\widehat{c}(\alpha)\star.\end{aligned}$$

The associated Dirac operator of this Clifford bundle is a first order elliptic differential operator D defined on the space of differential forms $\Omega(S^3)$ by $D := d + d^\dagger$ ([4, Proposition 3.53]). Here $d^\dagger$ denotes the formal adjoint of the exterior derivative d with respect to the L^2-inner product (3.1), which in this concrete case is $d^\dagger = \star d \star$. The operator $D = d + d^\dagger$ is called the **de Rham-Hodge operator**. Let us describe some of the most important properies of D:

- D is an elliptic first order differential operator, i.e. its principal symbols in invertible outside the zero section. Recall that the principal symbol of d and $d^\dagger$ are ([4, Proposition 2.1])

$$\sigma_P(d)(x,\xi) = -\,i\xi\wedge, \tag{3.4}$$

$$\sigma_P(d^\dagger)(x,\xi) = i\iota_{\xi^\sharp}, \tag{3.5}$$

  for $(x,\xi) \in T^*S^3$. Thus, the principal symbol of D is $\sigma_P(D)(x,\xi) = -ic(\xi)$, which is invertible for $\xi \neq 0$. Its inverse is just $-ic(\xi)/\|\xi\|^2$.
- Since S^3 is a closed manifold then D is a discrete essentially self-adjoint operator.
- As the dimension of S^3 is odd, then $D\star = \star D$.

3.2. **S^1-Invariant differential forms.** The S^1-action on S^3 induces an action on the vector bundle $\wedge T^*S^3 \otimes \mathbb{C}$ so that it becomes a S^1-bundle, i.e. the action on $\wedge T^*S^3 \otimes \mathbb{C}$ commutes with the Hopf map. The induced action on differential forms is just the pullback map, i.e.

$$\lambda \cdot \beta := (\lambda^{-1})^*\beta \quad \text{for } \lambda \in S^1 \text{ and } \beta \in \Omega(S^3). \tag{3.6}$$

Let $X = -e_3 = -(\partial_{\xi_1} + \partial_{\xi_2})$ be the generating vector field of the S^1 action. We call the associated dual 1-form $\chi := -e^3$ the **characteristic 1-form**. Let us define the space of **S^1-invariant forms** by

$$\Omega(S^3)^{S^1} := \{\beta \in \Omega(S^3) \mid L_X\beta = 0\}.$$

Since S^1 is connected, $\beta \in \Omega(S^3)^{S^1}$ if and only if $\lambda \cdot \beta = \beta$ for all $\lambda \in S^1$. In addition, we define the space of **basic forms** by

$$\Omega_{\text{bas}}(S^3) := \{\beta \in \Omega(S^3) \mid L_X\beta = 0 \text{ and } \iota_X\beta = 0\}.$$

An important characterization of basic forms is the following: A form $\beta \in \Omega(S^3)$ is basic if and only if there exists $\bar{\beta} \in \Omega(S^2)$ such that $\beta = \pi^*(\bar{\beta})$ ([17, Lemma 6.44]).

Remark 3.2 (Cohomology)**.** Note that both $(\Omega(S^3)^{S^1}, d)$ and $(\Omega_{\text{bas}}(S^3), d)$ are subcomplexes of the de Rham complex $(\Omega(S^3), d)$. In particular, we can define the S^1-invariant cohomology $H^*_{S^1}(S^3)$ (since S^1 is a compact Lie group there is an isomorphism $H^*_{S^1}(S^3) \cong H^*(S^3;\mathbb{R})$) and the basic cohomology $H^*_{\text{bas}}(S^3)$.

It is easy to verify, using the relation $L_X\chi = 0$, that the map

$$\begin{aligned} \Omega^k_{\text{bas}}(S^3) \oplus \Omega^{k-1}_{\text{bas}}(S^3) &\longrightarrow \Omega^k(S^3)^{S^1} \\ (\beta_0, \beta_1) &\longmapsto \beta_0 + \beta_1 \wedge \chi \end{aligned} \tag{3.7}$$

is an isomorphism. Moreover, there exists a short exact sequence of chain complexes ([23, Proposition 6.12])

$$0 \longrightarrow \Omega^*_{\text{bas}}(S^3) \hookrightarrow \Omega^*(S^3)^{S^1} \xrightarrow{\iota_X} \Omega^{*-1}_{\text{bas}}(S^3) \longrightarrow 0.$$

As a consequence of this decomposition, we will denote an S^1-invariant form $\beta_0 + \beta_1 \wedge \chi$ as

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}.$$

We define the **transversal Hodge star operator** $\bar{*} : \Omega^k_{\text{bas}}(S^3) \longrightarrow \Omega^{2-k}_{\text{bas}}(S^3)$ on the space of basic forms by the relations ([23, Chapter 7])

$$\bar{*}\beta = - * ((\varepsilon\beta) \wedge \chi) \quad \text{and} \quad * \beta = \bar{*}\beta \wedge \chi. \tag{3.8}$$

From this definition we see that $\text{vol}_{S^3} = *1 = \bar{*}1 \wedge \chi$. On the other hand,

$$\text{vol}_{S^3} = -e^1 \wedge e^2 \wedge e^3 = \text{vol}_{S^2} \wedge (-e^3),$$

so $\bar{*}1 = e^1 \wedge e^2$. If we denote the Hodge star operator and volume form on S^2 with respect to the metric (2.4) by $*_{S^2}$ and vol_{S^2} respectively, then following diagram commutes

$$\begin{tikzcd} \Omega^k_{\text{bas}}(S^3) \arrow[r, "\bar{*}"] & \Omega^{2-k}_{\text{bas}}(S^3) \\ \Omega^k(S^2) \arrow[u, "\pi^*"] \arrow[r, "*_{S^2}"] & \Omega^{2-k}(S^2) \arrow[u, "\pi^*"'] \end{tikzcd} \tag{3.9}$$

To see this observe,

$$\pi^*(\bar{\alpha} \wedge *_{S^2}\bar{\beta}) = \pi^*(\langle\bar{\alpha}, \bar{\beta}\rangle \text{vol}_{S^2}) \wedge \chi = \langle\pi^*(\bar{\alpha}), \pi^*(\bar{\beta})\rangle \pi^*(\text{vol}_{S^2}) \wedge = \langle\pi^*(\bar{\alpha}), \pi^*(\bar{\beta})\rangle \bar{*}1.$$

Similarly, we can also consider the **transversal cirality operator** ([12, Section 5])

$$\bar{\star} := (-1)^{k(k-1)/2} i\bar{*} = i^{k(k-1)+1}\bar{*}.$$

3.3. **Dirac Operator on S^1-Invariant forms.** As S^1acts on S^3 by orientation preserving isometries and since that the exterior derivative commutes with pullback we easily see that the operator D commutes with the action (3.6), i.e. $\forall g \in S^1$ and $\forall \beta \in \Omega(S^3)$ we have $D(g \cdot \beta) = g \cdot D\beta$. Hence, we can restrict D to the space $\Omega(S^3)^{S^1}$. Our aim is to compute this restriction operator with respect to the decomposition (3.7). This will allow us to "push-down" D to S^2 via π (This is the main idea of the work of Brüning and Heintze [7]).

3.3.1. *Decomposition of $\star$.* First we want to compute $\star$ in terms of the decomposition (3.7). We begin calculating for a basic differential form $\beta \in \Omega^k_{\text{bas}}(S^3)$,

$$\star\beta = -i^{k(k+1)} * \beta = -i^{k(k+1)} \bar{*}\beta \wedge \chi = i^{1+2k} i^{k(k-1)+1} \bar{*}\beta \wedge \chi = (i\varepsilon\bar{\star}\beta) \wedge \chi.$$

If we apply last formula to $i\varepsilon\bar{\star}\beta$ we obtain

$$\star(i\varepsilon\bar{\star}\beta) = -\beta \wedge \chi,$$

so using the relation$\star^2 = 1$ we get

$$\star^2(i\varepsilon\bar{\star}\beta) = (i\varepsilon\bar{\star}\beta) = - \star (\beta \wedge \chi).$$

Thus, we conclude that with respect to the decomposition (3.7) we can express

$$\star\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} 0 & -i\varepsilon\bar{\star} \\ i\varepsilon\bar{\star} & 0 \end{pmatrix}. \tag{3.10}$$

Note in particular that since $\varepsilon\bar{\star} = \bar{\star}\varepsilon$,

$$(i\varepsilon\bar{\star})^\dagger = -i\bar{\star}\varepsilon = -i\varepsilon\bar{\star}.$$

3.3.2. *Decomposition of d.* Recall that the space of basic differential forms defines a complex, thus the following diagram commutes:

$$\begin{tikzcd} \Omega^k_{\text{bas}}(S^3) \arrow[r, "d"] & \Omega^{k+1}_{\text{bas}}(S^3) \\ \Omega^k(S^2) \arrow[u, "\pi^*"] \arrow[r, "d_{S^2}"] & \Omega^{k+1}(S^2). \arrow[u, "\pi^*"'] \end{tikzcd} \tag{3.11}$$

Let us define the 2-form

$$\varphi := d\chi = 2e^1 \wedge e^2 = -i\Omega.$$

For $\beta_0 + \beta_1 \wedge \chi$, with $\beta_0, \beta_1 \in \Omega_{\text{bas}}(S^3)$ we calculate

$$\begin{aligned} d(\beta_0 + \beta_1 \wedge \chi) =& d\beta_0 + (d\beta_1) \wedge \chi + (\varepsilon\beta_1) \wedge d\chi \\ =& d\beta_0 + \varepsilon\varphi \wedge \beta_1 + (d\beta_1) \wedge \chi. \end{aligned}$$

Hence, we can write

$$d\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} d & \varepsilon\varphi\wedge \\ 0 & d \end{pmatrix}. \tag{3.12}$$

We use (3.10) to compute similarly for $d^\dagger = \star d \star$,

$$
\begin{aligned}
d^\dagger\Big|_{\Omega(M_0)^{S^1}} &= \begin{pmatrix} 0 & -i\varepsilon\bar{\star} \\ i\varepsilon\bar{\star} & 0 \end{pmatrix} \begin{pmatrix} d & \varepsilon\varphi\wedge \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & -i\varepsilon\bar{\star} \\ i\varepsilon\bar{\star} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i\varepsilon\bar{\star} \\ i\varepsilon\bar{\star} & 0 \end{pmatrix} \begin{pmatrix} i\varphi\wedge\bar{\star} & -id\varepsilon\bar{\star} \\ id\varepsilon\bar{\star} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i\varepsilon\bar{\star} \\ i\varepsilon\bar{\star} & 0 \end{pmatrix} \begin{pmatrix} i\varphi\wedge\bar{\star} & i\varepsilon d\bar{\star} \\ -i\varepsilon d\bar{\star} & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\bar{\star} d\bar{\star} & 0 \\ -\bar{\star}(\varepsilon\varphi\wedge)\bar{\star} & -\bar{\star} d\bar{\star} \end{pmatrix}.
\end{aligned}
$$

Thus, we conclude that under the decomposition (3.7) of $\Omega(S^3)^{S^1}$ we have

$$
S := D\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} d - \bar{\star} d\bar{\star} & \varepsilon(\varphi\wedge) \\ -\varepsilon\bar{\star}(\varphi\wedge)\bar{\star} & d - \bar{\star} d\bar{\star} \end{pmatrix}. \tag{3.13}
$$

From [7, Lemma 2.2] we know that the operator S is still essentially self-adjoint. Since the Hodge-de Rham operator D_{S^2} on S^2 is given by

$$
D_{S^2} = d_{S^2} + d^\dagger_{S^2} = d_{S^2} - \star_{S^2} d_{S^2} \star_{S^2},
$$

we see from (3.9) and (3.11) that S induces a self-adjoint operator ([7, Theorem 1.3]) defined sections of $\wedge T^*S^2 \otimes \mathbb{C}^2$ by the formula

$$
T := \begin{pmatrix} D_{S^2} & \varepsilon(\bar{\varphi}\wedge) \\ -\varepsilon \star_{S^2} (\bar{\varphi}\wedge)\star_{S^2} & D_{S^2} \end{pmatrix}
$$

where $\varphi = \pi^*(\bar{\varphi})$. This 2-form is explicitly given by

$$
\bar{\varphi} := \frac{1}{2}\sin\theta d\theta \wedge d\phi.
$$

Remark 3.3. We now give an interpretation to the operator $-\star_{S^2} (\bar{\varphi}\wedge)\star_{S^2}$. Recall that in S^2 we have $\star_{S^2}(\alpha\wedge)\star_{S^2} = \iota_{\alpha^\sharp}$ and $(\alpha\wedge)^\dagger = \iota_{\alpha^\sharp}$ (with respect to the L^2-inner product defined by the volume form vol_{S^2}) for $\alpha \in T^*S^2$. We can use these equations to compute

$$
\star_{S^2}(e^1 \wedge e^2\wedge)\star_{S^2} = \iota_{e_1} \star^2_{S^2} \iota_{e_2} = -\iota_{e_2}\iota_{e_1} = -(e^1 \wedge e^2\wedge)^\dagger.
$$

This shows the relation $-\star_{S^2} (\bar{\varphi}\wedge)\star_{S^2} = (\bar{\varphi}\wedge)^\dagger$, which allows us write

$$
T = \begin{pmatrix} D_{S^2} & \varepsilon(\bar{\varphi}\wedge) \\ \varepsilon(\bar{\varphi}\wedge)^\dagger & D_{S^2} \end{pmatrix}.
$$

Remark 3.4 (An Involution)**.** Since the dimension of S^2 is even, the operator D satisfies $D_{S^2} \star_{S^2} + \star_{S^2} D_{S^2} = 0$. Moreover, it is clear that $\star_{S^2}(\bar{\varphi}\wedge) + (\bar{\varphi})^\dagger \wedge \star_{S^2} = 0$, therefore

$$
T\vartheta + \vartheta T = 0 \tag{3.14}
$$

where ϑ is the self-adjoint involution

$$
\vartheta := \begin{pmatrix} 0 & \star_{S^2} \\ \star_{S^2} & 0 \end{pmatrix}.
$$

3.4. **A transversally elliptic Dirac-type operator.** Note that the induced operator T from (3.14) acts on two copies of $\Omega(S^2)\otimes\mathbb{C}$ instead of one. We will like to modify the above construction such that the induced operator acts on $\Omega(S^2)\otimes\mathbb{C}$ and such that the zero oder part anti-commutes with chirality operator $\star_{S^2}$ (why? Think about the index theorem). Motivated by the Operator S from (3.13) we define the operator ([20, Proposition 4.27])

$$B \coloneqq c(\chi)d - d^{\dagger}c(\chi).$$

Clearly B is a self-adjoint first order differential operator. Using (3.4) we can compute its principal symbol

$$\begin{aligned}\sigma_P(B)(x,\xi) =& ic(\chi)\xi\wedge + i\iota_{\xi^\sharp}c(\chi)\\ =& ic(\chi)\xi\wedge + i\langle\chi,\xi\rangle - ic(\chi)\iota_{\xi^\sharp}\\ =& ic(\chi)c(\xi) + i\langle\chi,\xi\rangle\end{aligned}$$

for $(x,\xi)\in T^*S^3$. Note that on one hand $\sigma_P(B)(x,\chi_x)=0$ and on the other if $\langle\chi,\xi\rangle=0$ then $\sigma_P(B)(x,\xi)=ic(\chi)c(\xi)$. For the later case $(ic(\chi)c(\xi))^2=\|\xi\|^2$ so we see that B is a transversally elliptic operator.

This operator, which is defined in terms of d and the Clifford multiplication $c(\chi)$, also commutes with the S^1-action (3.6). We now find an expression for B when restricted to S^1-invariant forms with respect to the decomposition (3.7). First note that with respect this decomposition we have

$$c(\chi)\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} 0 & -\varepsilon\\ \varepsilon & 0\end{pmatrix}.$$

Using (3.12) we compute

$$c(\chi)d\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} 0 & -\varepsilon\\ \varepsilon & 0\end{pmatrix}\begin{pmatrix} d & \varepsilon\varphi\wedge\\ 0 & d\end{pmatrix} = \begin{pmatrix} 0 & d\varepsilon\\ -d\varepsilon & \bar{\varphi}\wedge\end{pmatrix}.$$

Similarly

$$d^{\dagger}c(\chi)\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} -\bar{\star}d\bar{\star} & 0\\ -\varepsilon\bar{\star}(\varphi\wedge)\bar{\star} & -\bar{\star}d\bar{\star}\end{pmatrix}\begin{pmatrix} 0 & -\varepsilon\\ \varepsilon & 0\end{pmatrix} = \begin{pmatrix} 0 & \bar{\star}d\bar{\star}\varepsilon\\ -\bar{\star}d\bar{\star}\varepsilon & \bar{\star}(\varphi\wedge)\bar{\star}\end{pmatrix}.$$

Hence

$$B\Big|_{\Omega(S^3)^{S^1}} = \begin{pmatrix} 0 & (d-\bar{\star}d\bar{\star})\varepsilon\\ -(d-\bar{\star}d\bar{\star})\varepsilon & (\varphi\wedge - \bar{\star}(\varphi\wedge)\bar{\star})\end{pmatrix}. \tag{3.15}$$

Observe that operator B satisfies $B\varepsilon=\varepsilon B$ so it can be decomposed as $B=B_{\text{ev}}\oplus B_{\text{odd}}$ where $B_{\text{ev/odd}}:\Omega^{\text{ev/odd}}(S^3)\longrightarrow\Omega^{\text{ev/odd}}(S^3)$. Each component of this operator obviously commutes with the S^1-action, so we can consider the operator

$$B^{\text{ev}}:\Omega^{\text{ev}}(S^3)^{S^1}\longrightarrow\Omega^{\text{ev}}(S^3)^{S^1}.$$

Consider the unitary transformations ([8, Section 5])

$$\begin{array}{rcl} \psi_k: \ \Omega^{k-1}(S^2) \oplus \Omega^k(S^2) & \longrightarrow & \Omega^k(S^3)^{S^1} \\ (\beta_{k-1}, \beta_k) & \longmapsto & \pi^*(\beta_k) + \pi^*(\beta_{k-1}) \wedge \chi \end{array}.$$

for $k = 0, 1, 2, 3$ (here $\Omega^{-1}(S^2) := \{0\}$). Using the notation $\beta_k \in \Omega^k(S^2)$ we define two further unitary transformations

$$\begin{array}{rcl} \psi_{\mathrm{ev}}: \quad \Omega(S^2) & \longrightarrow & \Omega^{\mathrm{ev}}(S^3)^{S^1} \\ (\beta_0, \beta_1, \beta_2) & \longmapsto & (\psi_0(0, \beta_0), \psi_2(\beta_1, \beta_2)), \end{array}$$

$$\begin{array}{rcl} \psi_{\mathrm{odd}}: \quad \Omega(S^2) & \longrightarrow & \Omega^{\mathrm{odd}}(S^3)^{S^1} \\ (\beta_0, \beta_1, \beta_2) & \longmapsto & \psi_1(\beta_0, \beta_1). \end{array}$$

We want to compute the operator $\mathscr{D}_{S_2} := \psi_{\mathrm{ev}} B \psi_{\mathrm{odd}}^{-1} : \Omega(S^2) \longrightarrow \Omega(S^2)$, i.e. the operator $\mathscr{D}$ that fits into the commutative diagram

$$\begin{tikzcd} \Omega^{\mathrm{ev}}(S^3)^{S^1} \arrow[r, "B_{\mathrm{ev}}"] & \Omega^{\mathrm{odd}}(S^3)^{S^1} \\ \Omega(S^2) \arrow[u, "\psi_{\mathrm{odd}}"] \arrow[r, "\mathscr{D}_{S^2}"] & \Omega(S^2). \arrow[u, "\psi_{\mathrm{ev}}"'] \end{tikzcd}$$

If we define the endomorphism

$$\widehat{c}(\bar{\varphi}) := \bar{\varphi} \wedge + (\bar{\varphi} \wedge)^\dagger = \bar{\varphi} \wedge - \bar{\star}(\bar{\varphi} \wedge)\bar{\star},$$

which is a kind of right Clifford multiplication by $\bar{\varphi}$, we see from (3.15) that

$$\mathscr{D}_{S^2} = D_{S^2} + \widehat{c}(\bar{\varphi}). \tag{3.16}$$

It is important to see that since $\widehat{c}(\bar{\varphi})\bar{\star} + \bar{\star}\widehat{c}(\bar{\varphi}) = 0$, then

$$\mathscr{D}_{S^2}\bar{\star} + \bar{\star}\mathscr{D}_{S^2} = 0.$$

This means that we have found an induced operator $\mathscr{D}_{S^2}$ which is of Dirac-Schrödinger type, whose first order part is the Hodge-de Rham operator D_{S^2} on S^2 and such that it anti-commutes with $\star_{S^2}$ ([20, Theorem 4.28]). Hence we can decompose $\mathscr{D}_{S^2}$ with respect to $\star_{S^2}$ as

$$\mathscr{D}_{S^2} = \begin{pmatrix} 0 & \mathscr{D}_{S^2,+} \\ \mathscr{D}_{S^2,-} & 0 \end{pmatrix}.$$

Observe that $\mathrm{ind}(D_{S^2,+}) = 0$ by the signature theorem ([4, Theorem 4.8]), so therefore $\mathrm{ind}(\mathscr{D}_{S^2,+}) = \mathrm{ind}(D_{S^2,+}) = 0$ since the index just depends on the principal symbol.

As an important remark, note that even though in this concrete example the index vanishes, when one goes to higher dimensions the induced operator from B will produce non-trivial indices. Moreover, the action need not be free since one can consider the action on the principal orbit. This procedure, inspired in [7], allow us to produce from the operator B self-adjoint operators on singular spaces arising from S^1-actions which anti-commute with the signature grading ([19], [20]).

4. Spin-Dirac operator

In this section we want to proceed similarly as we did before but in the context of the spin-Dirac operator. First we will show how to construct the spinor bundle and the spin-Dirac operator on S^2 and S^3. We also show how they are related trough the Hopf map when resctricted to S^1-invariant spinors. As before, we do not aim to explain the theory in detail but rather focus on the computations. Standard references on this topic are [4], [15], [18],[21] and [22] among many others.

4.1. **The spin Dirac operator for S^2.**

4.1.1. *Spin strucrure.* We begin by giving an explicit construction of the unique spin structure on S^2. First recall that $\text{Spin}(2) = SO(2) = S^1$ and that there is a short exact sequence of groups

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow S^1 \xrightarrow{\varrho} S^1 \longrightarrow 0,$$

where

$$\begin{aligned} \varrho : \ S^1 &\longrightarrow S^1 \\ \lambda &\longmapsto \lambda^2. \end{aligned}$$

Let us denote by $\pi_{so} : SO(S^2) \longrightarrow S^2$ the oriented frame bundle of S^2, which is an $SO(2)$-principal bundle. A **spin structure** on S^2 consists of a principal Spin(2)-bundle $\pi_{spin} : \text{Spin}(S^2) \longrightarrow S^2$ and a 2-fold covering map $q : \text{Spin}(S^2) \longrightarrow SO(S^2)$ such that the following diagram commutes

$$\begin{tikzcd} \text{Spin}(S^2) \times S^1 \arrow[r] \arrow[d, "q\times\varrho"] & \text{Spin}(S^2) \arrow[d, "q"] \arrow[dr, "\pi_{spin}"] & \\ SO(S^2) \times S^1 \arrow[r] & SO(S^2) \arrow[r, "\pi_{so}"'] & S^2 \end{tikzcd} \tag{4.1}$$

A topological condition for the existence of spin structures is the vanishing of the second Stiefel-Whitney class $w_2(TS^2) \in H^2(S^2; \mathbb{Z}/2\mathbb{Z})$ ([15, Theorem II.1.7]). This characteristic class is the $\mathbb{Z}/2\mathbb{Z}$-reduction of the Euler class of TS^2 ([16]). By the Gauß-Bonnet theorem we know that the integral of this Euler class equals the Euler characteristic $\chi(S^2) = 2$, which modulo $\mathbb{Z}/2\mathbb{Z}$ is zero. This shows that S^2 is a spin manifold. Moreover, the spin structures are classified by the group $H^1(S^2; \mathbb{Z}/2\mathbb{Z}) = 0$, so we conclude that S^2 has only one spin structure. This result is actually valid for all spheres.

Our first objective is to show that the Hopf bundle defines the unique spin structure on S^2. We begin by describing the frame bundle $SO(S^2)$. We claim that $SO(S^2) = SO(3)$. More precisely consider the action of $SO(3)$ on S^2 by rotations. Since this action is transitive and the isotropy group of any point on the sphere is $SO(2)$ we see that $SO(3)/SO(2) \cong S^2$ and we have a $SO(2)$-principal bundle $SO(3) \longrightarrow S^2$. On the other hand let us see how can a rotation $R \in SO(3)$ define an frame for some point in the sphere. The rotation R is characterized by the rotation axis, which can be defined trough a unit vector, and the rotation angle. The unite vector defined the point on the sphere and the angle defined the orthonormal basis on the tangent space of this point.

Now that we have described the frame bundle we will see how the Hopf fibration fits into the diagram (4.1). Recall that we can identify $SO(3) \cong \mathbb{R}P^3$ and denote by $q : S^3 \longrightarrow \mathbb{R}P^3$ the antipodal map, i.e.

$$
\begin{aligned}
q: \quad & S^3 \longrightarrow \mathbb{R}P^3 \\
& (z_0, z_1) \longmapsto [z_0, z_1],
\end{aligned}
$$

where $[z_0, z_1] = [-z_0, -z_1]$. Note that the Hopf map (2.1) satisfies $\pi(z_0, z_1) = \pi(-z_0, -z_1)$ so it defines a map $\pi_{so} : \mathbb{R}P^3 \longrightarrow S^2$. However, the principal structure action is given by the S^1-action $\lambda \cdot [z_0, z_1] := [\lambda^{1/2} z_0, \lambda^{1/2} z_1]$. Note that the sign of the square root is irrelevant since the coordinates on $\mathbb{R}P^3$ are invariant under a change of sign. Hence we have constructed the commutative diagram

$$
\begin{tikzcd}
S^3 \times S^1 \arrow[r] \arrow[d, "q \times \varrho"] & S^3 \arrow[d, "q"] \arrow[dr, "\pi"] & \\
\mathbb{R}P^3 \times S^1 \arrow[r] & \mathbb{R}P^3 \arrow[r, "\pi_{so}"'] & S^2.
\end{tikzcd}
$$

From the diagram (4.1) we see how the Hopf fibration defines the unique spin structure on S^2.

4.1.2. *The spinor bundle.* Now that we have studied the spin structure explicitly we will construct the spinor bundle $\Sigma(S^2)$ as an associated bundle $\Sigma(S^2) = \text{Spin}(S^2) \times_{\rho_2} \Sigma_2$ where $\rho_2 : S^1 \longrightarrow \text{Aut}(\Sigma_2)$ is the spin representation and $\Sigma_2 = \mathbb{C}^2$ is the spinor space. In particular we sill show that it is a trivial rank 2 complex vector bundle.

To begin with we describe the spin representation of the Clifford algebra $Cl(2)$ on the spinor vector space $\Sigma_2 = \mathbb{C}^2$. To to so its enough to describe the action on basis elements. Recall the definition of the **Pauli matrices**

$$
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

These matrices satisfy the relations

$$\begin{aligned}\sigma_j^\dagger =& \sigma_j,\\ \sigma_j^2 =& 1,\\ \sigma_1\sigma_2 =& i\sigma_3,\\ \sigma_j\sigma_k + \sigma_k\sigma_j =& 2\delta_{jk} \quad \text{for } j = 1, 2, 3.\end{aligned}$$

Let $\{v_1, v_2\}$ be the standard orthonormal basis of $\mathbb{R}^2$, we define the Clifford action $\rho_2(v_j) := -i\sigma_j$ for $j = 1, 2$. It follows that

$$\rho_2(v_j)\rho_2(v_k) + \rho_2(v_k)\rho_2(v_j) = -2\delta_{jk}.$$

We now want to study the restriction of this representation to $\mathrm{Spin}(2) \subset Cl(2)$. Every element of Spin(2) can be written as

$$\cos t + \sin t v_1 v_2 = -(\sin(t/2)v_1 + \cos(t/2)v_2)(\cos(t/2)v_1 + \sin(t/2)v_2),$$

for $t \in [0, 2\pi]$, so

$$\rho_2(\cos t + \sin t v_1 v_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos t + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \sin t = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}.$$

This shows that the spin representation restricted to $\mathrm{Spin}(2) = S^1$ is given by

$$\begin{aligned}\rho_2 :\ & S^1 \longrightarrow \mathrm{Aut}(\Sigma_2)\\ & z \longmapsto \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix}.\end{aligned} \tag{4.2}$$

We now compute the transition functions of the spinor bundle $\Sigma(S^2) = \mathrm{Spin}(S^2) \times_{\rho_2} \Sigma_2$. These are obtained by composing the transition functions of the Hopf bundle with ρ_2, i.e. for $\pi(z_0, z_1) \in U_N \cap U_S$ we have

$$\rho_2(\pi(z_0, z_1)) = \rho_2\left(\frac{z_0}{z_1}\right) = \begin{pmatrix} z_0/z_1 & 0 \\ 0 & \bar{z}_0/\bar{z}_1 \end{pmatrix}.$$

We will now explain how to see that the spinor bundle $\Sigma(S^2)$ is trivial. Our argument goes into the direction of clutching functions in K-theory. Let $\mathrm{Vect}^k_{\mathbb{C}}(S^2)$ denote the monoid of isomorphims classes of complex vector bundles of rank k over S^2. An important result in the context of classification of vector bundles states that the map $\Phi : [S^1, GL(k, \mathbb{C})] \longrightarrow \mathrm{Vect}^k_{\mathbb{C}}(S^2)$ defined by the transition functions (clutching functions) is a bijection ([13, Proposition1.11]). Moreover, as groups we have an isomorphism $[S^1, GL(k, \mathbb{C})] \cong [S^1, U(k, \mathbb{C})] \cong \mathbb{Z}$, the first one given by the Gram-Schmidt orthogonalization process and the second one. One also has that the map $\det : [S^1, U(k, \mathbb{C})] \longrightarrow S^1$ is well defined and the degree of this map $\deg \circ \det : [S^1, U(k, \mathbb{C})] \longrightarrow \mathbb{Z}$ is actually an isomorphism. This means that the degree of the determinant of the clutching functions characterizes the isomorphism class of the associated vector bundle. This argument shows that since

$$\det \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} = \bar{z}z = 1,$$

for $z \in S^1$, we see that the associated bundle must be isomorphic to the trivial bundle. One can actually define an explicit homotopy ([2, Section 2.4]) for $t \in [0, \pi/2]$,

$$\gamma(t) := \begin{pmatrix} \bar{z} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Observe that $\det(\gamma(t)) = 1$ for all $t \in [0, \pi/2]$ and

$$\gamma(0) = \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} \quad \text{and} \quad \gamma(\pi/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the sections of $\Sigma(S^2)$, called **spinors**, can be regarded as functions $\psi : S^2 \longrightarrow \mathbb{C}^2$.

One can also show the triviality of $\Sigma(S^2)$ by describing it in term of the exterior bundle and comparing how the sections transform. We refer to [24, Section A.2].

4.1.3. *The spin-Dirac operator.* We now want to construct the Dirac operator $\not{D}_{S^2(r)}$ associated to the Clifford bundle $\Sigma(S^2)$. Here we consider the 2-sphere of radius $r > 0$ in view of the fact that we will need it for $r = 1/2$ when we study the induced spin-Dirac operator of the Hopf fibration. The corresponding **spin connection** ∇^Σ can be computed using the component of the connection 1-form (1.5) as [15, Theorem 4.14]

$$\nabla^\Sigma = \frac{1}{2}\omega_{21} \otimes \rho_2(e_1)\rho_2(e_2) = -\frac{1}{2}\omega_{12} \otimes \sigma_2\sigma_1 = -\frac{i}{2}\frac{\cot\theta}{r}e^2 \otimes \sigma_3.$$

The corresponding Dirac operator is then

$$\not{D}_{S^2(r)} := \rho_2(e_1)\nabla^\Sigma_{e_1} + \rho_2(e_2)\nabla^\Sigma_{e_2}.$$

From the explicit expression of the spin connection he have

$$\begin{aligned} -i\sigma_1\nabla^\Sigma_{e_1} &= - i\sigma_1 e_1, \\ -i\sigma_2\nabla^\Sigma_{e_2} &= - i\sigma_2 e_2 - \frac{1}{2}\frac{\cot\theta}{r}\sigma_2\sigma_3 = -i\sigma_2 e_2 - \frac{i}{2}\frac{\cot\theta}{r}\sigma_1, \end{aligned}$$

so he obtain

$$\not{D}_{S^2(r)} = -i\sigma_1\nabla^\Sigma_{e_1} - i\sigma_2\nabla^\Sigma_{e_2} = -i\sigma_1\left(e_1 + \frac{1}{2}\frac{\cot\theta}{r}\right) - i\sigma_2 e_2. \tag{4.3}$$

In terms of the coordinate vector fields we can write

$$\not{D}_{S^2(r)} = -i\sigma_1\left(\frac{1}{r}\partial_\theta + \frac{1}{2}\frac{\cot\theta}{r}\right) - \frac{i\sigma_2}{\sin\theta}\partial_\phi.$$

Note from the relation for $f_1, f_2 \in C^\infty(S^2)$,

$$\frac{d}{d\theta}(f_1 f_2 \sin\theta) = (f_1' f_2 + f_1 f_2' + f_1 f_2 \cot\theta)\sin\theta,$$

that $\not{D}_{S^2(r)}$ satisfies

$$\int_{S^2} \langle \not{D}_{S^2}\psi_1, \psi_2\rangle_{\mathbb{C}^2} r^2 \sin\theta d\theta \wedge d\phi = \int_{S^2} \langle \psi_1, \not{D}_{S^2}\psi_2\rangle_{\mathbb{C}^2} r^2 \sin\theta d\theta \wedge d\phi,$$

i.e. $\not{D}_{S^2(r)}$ is symmetric, and since $S^2(r)$ is closed then $\not{D}_{S^2(r)}$ is also essentially self-adjoint.

The chirality operator of associated to the spinor bundle $\Sigma(S^2)$ is ([4, Lemma 3.17])

$$\Gamma = i(-i\sigma_1)(-i\sigma_2) = \sigma_3. \tag{4.4}$$

Since the dimension of S^2 is even we can verify that $\not{D}_{S^2(r)}\sigma_3 + \sigma_3\not{D}_{S^2(r)} = 0$.

4.1.4. *Tautological bundle.* In this section we are going to give an alternative description of the spinor bundle $\Sigma(S^2)$ and in particular to the subbundles $\Sigma^{\pm}(S^2)$ ([3, Beispiel 2.11][24, Section A.2]). First let us recall how to identify S^2 with $\mathbb{C}P^1$ ([3, Beispiel 2.7]). Define locally two functions

$$z(\theta,\phi) := e^{i\phi}\tan(\theta/2) \quad \text{for } x(\theta,\phi) \in U_N, \tag{4.5}$$

$$w(\theta,\phi) := e^{-i\phi}\cot(\theta/2) \quad \text{for } x(\theta,\phi) \in U_S \tag{4.6}$$

Is it easy to see that (U_N, z) and (U_S, w) define two compatible charts on S^2 with coordinate transformation $z = 1/w$ on $U_N \cap U_S$. We now construct a diffeomorphism $\varrho : S^2 \longrightarrow \mathbb{C}P^1$ by defining it locally as (see (2.3))

$$\begin{aligned} \varrho_N : \quad U_N &\longrightarrow \mathbb{C}P^1 \\ x(\theta,\phi) &\longmapsto [z(\theta,\phi) : 1], \end{aligned}$$

$$\begin{aligned} \varrho_S : \quad U_S &\longrightarrow \mathbb{C}P^1 \\ x(\theta,\phi) &\longmapsto [1 : w(\theta,\phi)]. \end{aligned}$$

Here the homogeneous coordinates satisfy $[z : w] = [\mu z : \mu w]$ for every $\mu \in \mathbb{C} - \{0\}$. Observe that ϱ is defined globally since for $x(\theta,\phi) \in U_N \cap U_S$ we have

$$\varrho_N(x(\theta,\phi)) = [z(\theta,\phi) : 1] = [1/w(\theta,\phi) : 1] = [1 : w(\theta,\phi)] = \varrho_S(x(\theta,\phi)).$$

We now define the **tautological line bundle** $\Theta : \mathcal{L} \longrightarrow \mathbb{C}P^1$ locally as

$$\begin{aligned} \mathcal{L}_N &:= \{([z(\theta,\phi) : 1], \mu z(\theta,\phi)) \in \mathbb{C}P^1 \times \mathbb{C} \mid x(\theta,\phi) \in U_N,\ \mu \in \mathbb{C}\}, \\ \mathcal{L}_S &:= \{([1 : z(\theta,\phi)], \mu w(\theta,\phi)) \in \mathbb{C}P^1 \times \mathbb{C} \mid x(\theta,\phi) \in U_S,\ \mu \in \mathbb{C}\}. \end{aligned}$$

with local trivializations

$$\begin{aligned} \psi_N : \quad \Theta^{-1}(U_N) &\longrightarrow U_N \times \mathbb{C} \\ ([z(\theta,\phi) : 1], \mu z(\theta,\phi)) &\longmapsto (x(\theta,\phi), \mu) \end{aligned}$$

$$\begin{aligned} \psi_S : \quad \Theta^{-1}(U_S) &\longrightarrow U_S \times \mathbb{C} \\ ([1 : w(\theta,\phi)], \mu w(\theta,\phi)) &\longmapsto (x(\theta,\phi), \mu) \end{aligned}$$

We now compute the transition function explicitly. For $x(\theta,\phi) \in U_N \cap U_S$ we compute

$$\begin{aligned}
\psi_N \circ \psi_S^{-1}(x(\theta,\phi),\mu) =& \psi_N([1 : w(\theta,\phi)], \mu w(\theta,\phi)) \\
=& \psi_N([1/w(\theta,\phi) : 1], (w(\theta,\phi)\mu)1/w(\theta,\phi)) \\
=& \psi_N([z(\theta,\phi) : 1], ((1/z(\theta,\phi))\mu)z(\theta,\phi)) \\
=& (x(\theta,\phi), (1/z(\theta,\phi))\mu)).
\end{aligned}$$

Note that for $x(\theta,\phi) \in U_N \cap U_S$ we must have $\theta = \pi/2$, then (4.5) implies that $z(\phi,\pi/2) \in S^1$. We conclude then that transition function of the line bundle $\mathcal{L}$ with respect to the above trivialization is

$$h_{NS}(z) = 1/z = \bar{z}.$$

Hence, in view of (4.2) and (4.4), we see that as complex line bundles we have $\mathcal{L} \cong \Sigma^+(S^2)$, $\mathcal{L}^* \cong \Sigma^-(S^2)$ and $\mathcal{L} \oplus \mathcal{L}^* \cong \Sigma(S^2)$. From the later relation we see that

$$c(\Sigma(S^2)) = c(\mathcal{L} \oplus \mathcal{L}^*) = c_1(\mathcal{L}) + c_1(\mathcal{L}^*) = c_1(\mathcal{L}) - c_1(\mathcal{L}) = 0$$

as we expected since we have see that $\Sigma(S^2)$ is trivial.

4.2. **The spin-Dirac operator for S^3.** In this section we compute the spin-Dirac operator on S^3 with respect to the coordinates (1.13). In particular we will decompose it as an "horizontal" part which can be interpreted as the spinpin-Dirac operator on S^2, a "vertical" part and a zero order term ([1]). Hitchin computed in [14, Proposition 3.1] an explicit expression for the spin-Dirac operator on S^3 for more general metrics.

Since $S^3 \cong SU(2)$ has a Lie group structure it has trivial tangent bundle parallelized by a basis of its Lie algebra. This parallelization can be use to paralellize the spinor bundle ([14, Section 3.1]), i.e. $\Sigma(S^3) = S^3 \times \Sigma_2$.

We now go directly to the computation of the spin-Dirac operator. From (1.15) we see that the associate spin connection with respect to the basis (1.13) is ([15, Theorem 4.14])

$$\begin{aligned}
\nabla^\Sigma =& -\frac{1}{2}\omega_{12} \otimes \sigma_2\sigma_1 - \frac{1}{2}\omega_{13} \otimes \sigma_3\sigma_1 - \frac{1}{2}\omega_{23} \otimes \sigma_3\sigma_2 \\
=& -\frac{1}{2}(-2\cot 2\eta e^2 + e^3) \otimes (-i\sigma_3) - \frac{1}{2}e^2 \otimes (i\sigma_2) - \frac{1}{2}(-e^1) \otimes (-i\sigma_1) \\
=& -\frac{i}{2}(2\cot 2\eta e^2 - e^3) \otimes \sigma_3 - \frac{i}{2}e^2 \otimes \sigma_2 - \frac{i}{2}e^1 \otimes \sigma_1.
\end{aligned}$$

As before the Dirac operator is given by

$$\not{D}_{S^3} = -i\sigma_1 \nabla^\Sigma_{e_1} - i\sigma_2 \nabla^\Sigma_{e_2} - i\sigma_3 \nabla^\Sigma_{e_3}.$$

We compute separately each term

$$
\begin{aligned}
-i\sigma_1\nabla^{\Sigma}_{e_1} &= -\, i\sigma_1 e_1 - \frac{1}{2},\\
-i\sigma_2\nabla^{\Sigma}_{e_2} &= -\, i\sigma_2 e_2 - i\cot 2\eta\sigma_1 - \frac{1}{2},\\
-i\sigma_3\nabla^{\Sigma}_{e_3} &= -\, i\sigma_3 e_3 + \frac{1}{2},
\end{aligned}
$$

and conclude that

$$
\not{D}_{S^3} = -i\sigma_1(e_1 + \cot 2\eta) - i\sigma_2 e_2 - i\sigma_3 e_3 - \frac{1}{2}. \tag{4.7}
$$

The chirality operator Γ_{S^3} of the spinor bundle is ([4, Lemma 3.17])

$$
\Gamma_{S^3} = -i^2 c(e_1)c(e_2)c(e_3) = (-i\sigma_1)(-i\sigma_2)(-i\sigma_3) = -(i\sigma_3)(-i\sigma_3) = -1.
$$

Since the dimension of S^3 is odd we verify that $\not{D}_{S^3}\Gamma = \Gamma\not{D}_{S^3}$.

Now we want to study the construction of [7] for the free S^1-action on S^3 which defines the Hopf fibration. First of all note that since the S^1-action preserves the orientation and the metric then we can lift it to the principal frame bundle $SO(S^3) \longrightarrow S^3$. Since S^2 is a spin manifold, the action on $SO(S^3)$ lifts to the spin bundle $\mathrm{Spin}(S^3) \longrightarrow S^3$, i.e. the action is **projectable** ([1, Section 4], [5, Proposition 2.2]). Moreover, this action is inherited by the spinor bundle $\Sigma(S^3)$ such that it becomes and S^1-vector bundle. One can also show that the spin-Dirac operator $\mathscr{D}_{S^3}$ commutes with the S^1-action on spinors given by

$$
(\lambda \cdot \psi)(x) := \psi(\lambda^{-1} \cdot x), \tag{4.8}
$$

for $x \in S^3$ ([22, Section 19]). Recall that $X = -e_3$ denotes the generating vector field of the S^1-action, then it is easy to see the space of S^1-invariant spinors is with respect to the action (4.8) is

$$
\Gamma(S^3, \Sigma(S^3))^{S^1} = \{\psi \in \Gamma(S^3, \Sigma(S^3)) \mid L_X\psi = 0\}, \tag{4.9}
$$

where the Lie derivative $L_X\psi$ is defined by [4, Equation (1.2)]. Nevertheless, in this case this Lie derivative is just the Lie derivative each components. Is it easy to see that we can identify the spaces

$$
\Gamma(S^3, \Sigma(S^3))^{S^1} = \pi^*\Gamma(S^2, \Sigma(S^2)), \tag{4.10}
$$

that means that for each $\psi \in \Gamma(S^3, \Sigma(S^3))^{S^1}$ there exists a unique $\bar{\psi} \in \Gamma(S^2, \Sigma(S^2))$ such that $\psi(x) = \bar{\psi}(\pi(x))$ for all $x \in S^3$.

Observe now from (4.3), with $r = 1/2$, (4.7), (4.9), and (4.10) that the spin-Dirac operator restricted to the space of S^1-invariant sections is

$$
\not{D}_{S^3}\Big|_{\Gamma(S^3,\Sigma(S^3))^{S^1}} = \pi^*\not{D}_{S^2} - \frac{1}{2}. \tag{4.11}
$$

Remark 4.1. The decomposition (4.7) agrees with the decomposition derived in the proof of [1, Theorem 4.1]. They sow in a more general case how to decompose the spin-Dirac operator as a sum of a vertical part, an horizontal part and a zero order term. This last term is constructed from the connection form $\omega = -ie^3$ and the curvature $\Omega = 2ie^1 \wedge e^2$ of the principal S^1-bundle (Section 2.4). Using the fact that $\omega(ie_3) = 1$, the concrete expression derived in [1, pg. 240] is

$$-\frac{1}{4}c(ie_3)(2ic(e_1)c(e_2)) = -\frac{1}{4}i(-i\sigma_3)2i(-i\sigma_1)(-i\sigma_2) = \frac{1}{2}\sigma_3(i\sigma_1\sigma_2) = \frac{1}{2}\sigma_3(-\sigma_3) = -\frac{1}{2},$$

which coincides with (4.7).

Finally we see that form (4.11) that the operator induced on the spinor bundle of S^2 is

$$\not{T} := \not{D}_{S^2} - \frac{1}{2}. \tag{4.12}$$

Note however that the potential term in (4.12) is just multiplication by $-1/2$ and it does not anti-commute with the chirality operator (4.4).

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Institut Für Mathematik , Humboldt Universität zu Berlin, Germany.
Email address: juanitorduz@gmail.com