Induced Dirac-Schrödinger operators on S^1-semi-free quotients

Disputation der Dissertation von

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zur Erlangung des akademischen Grades Dr. rer. nat.

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The signature operator on a manifold with a conical singular stratum

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- Index computation (Witt case)

The induced operator Dirac-Schrödinger operator

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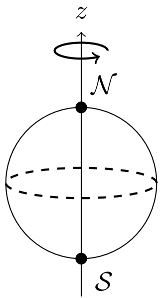
- Definition of $\mathscr{D}$
- Example revisited
- Local description of the potential
- Index computation (Witt case)
- Index computation (non-Witt case)?

Example: What is a semi-free action?

Let S^1 act on $M = S^2 \subset \mathbb{R}^3$ by rotations around the z-axis.

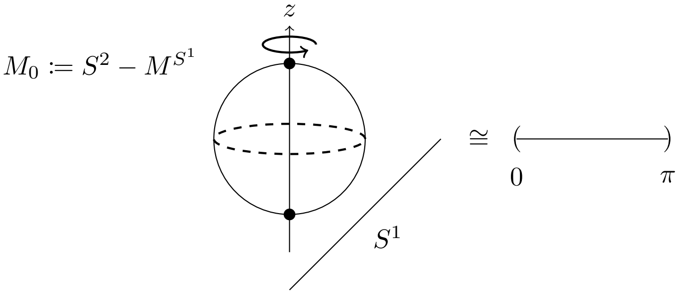
- $M^{S^1} := \{\mathcal{N}, \mathcal{S}\}$ fixed point set.
- On the principal orbit $M_0 := S^2 - M^{S^1}$, the action is free.



For a semi-free action the isotropy groups $S^1_x := \{g \in S^1 \mid gx = x\}$ must be either $\{1\}$ or S^1 for all $x \in S^2$.

Example: A S^1-semi-free quotient

As the action on M_0 is free, the quotient space $M_0/S^1 = (0, \pi) =: I$ is a smooth manifold.



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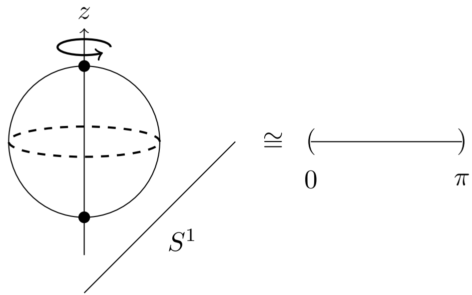

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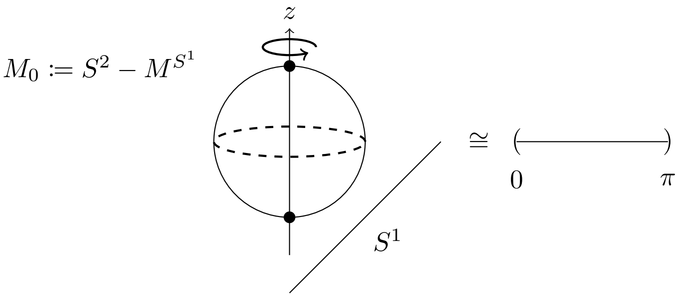

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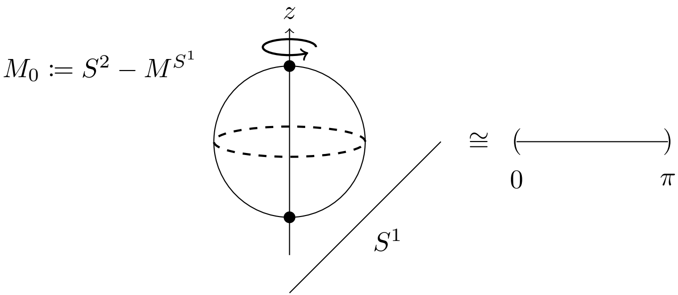

- Equip S^2 with the round metric $g^{TS^2} = d\theta^2 + \sin^2\theta d\phi^2$.
- The quotient metric $g^{TI} := d\theta^2$ is incomplete.
- The Hodge - de Rham operator

$$D_I := \begin{pmatrix} 0 & -\partial_\theta \\ \partial_\theta & 0 \end{pmatrix}$$

defined on $\Omega_c(I)$, is not essentially self-adjoint in $L^2(\wedge^* I)$.

Lott's S^1-equivariant signature

- (M, g^{TM}): $4k+1$-dimensional closed oriented Riemannian manifold on which S^1 acts by orientation preserving isometries.
- M^{S^1}: fixed point set and M_0 the principal orbit.
- V: generating vector field of the action.
- $\alpha := V^\flat/\|V\|^2 \in \Omega^1(M_0)$ satisfies $\alpha(V) = 1$.

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$$\Omega_{\mathrm{bas},c}(M_0) := \{\omega \in \Omega_c(M_0) \mid L_V\omega = 0 \text{ and } \iota_V\omega = 0\}.$$

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Define $\sigma_{S^1}(M)$ to be the signature of the symmetric quadratic form,

$$\begin{aligned} H^{2k}_{\mathrm{bas},c}(M_0) \times H^{2k}_{\mathrm{bas},c}(M_0) &\longrightarrow \mathbb{R} \\ ([\omega],[\omega']) &\longmapsto \int_M \alpha \wedge \omega \wedge \omega'. \end{aligned}$$

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- It does not depend on the Riemannian metric.
- It is invariant under S^1-homotopy equivalences.

$\sigma_{S^1}(M)$ formula for semi-free actions

Theorem (Lott, 00')

Suppose S^1 acts effectively and semi-freely on M, then

$$\sigma_{S^1}(M) = \int_{M_0/S^1} L(T(M_0/S^1), g^{T(M_0/S^1)}) + \eta(M^{S^1}).$$

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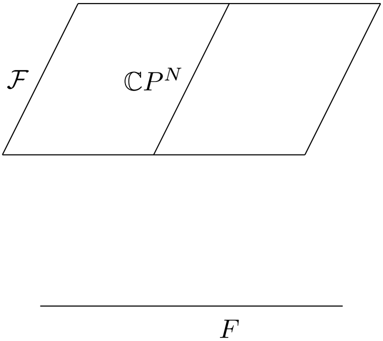
If the codimension of M^{S^1} in M is divisible by four we call M/S^1 a Witt space and

- $\eta(M^{S^1}) = 0.$
- $L(T(M_0/S^1), g^{T(M_0/S^1)})$ represents the homology L-class of M/S^1.
- $\sigma_{S^1}(M)$ equals the intersection homology signature of M/S^1.

Strategy of Lott's proof

- $F \subset M^{S^1}$: connected closed $4k - 2N - 1$ dimensional manifold.
- Let $NF \longrightarrow F$ be the normal bundle of F.
- SNF/S^1 is the total space of a Riemannian $\mathbb{C}P^N$-fiber bundle $\mathcal{F}$.
- Model M/S^1 as the mapping cylinder $C(\pi_{\mathcal{F}} : \mathcal{F} \longrightarrow F)$.
- For $r > 0$ small enough $\sigma_{S^1}(M) = \sigma(M/S^1 - N_r(F))$.
- Study the limit of the APS signature theorem as $r \longrightarrow 0$.
  - Use Dai's formula for the adiabatic limit of the eta invariant.
  - Prove that the form $\widetilde{\eta}$ and Dai's tau invariant $\tau_{\mathcal{F}}$ vanish.



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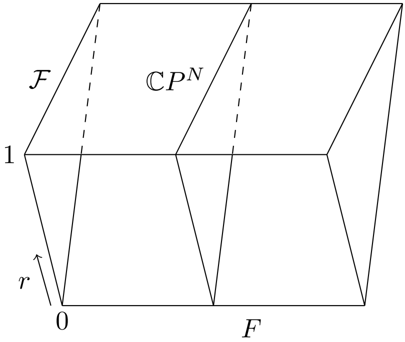

$$dr^2 \oplus g^{T_H\mathcal{F}} \oplus r^2 g^{T_V\mathcal{F}}$$

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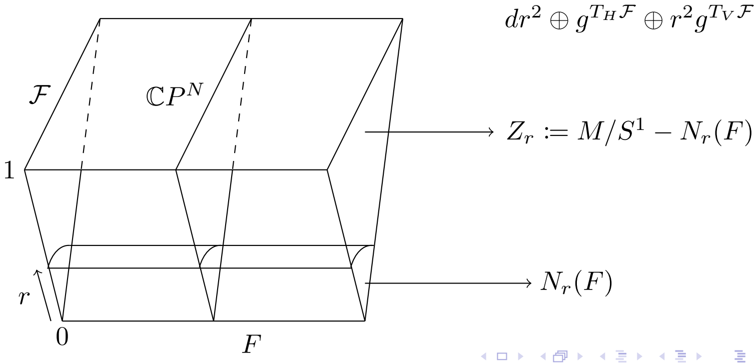

Does there exist an operator whose index is $\sigma_{S^1}(M)$?

Natural Candidate: the signature operator on M_0/S^1

- Consider the de Rham complex:

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- $\mathrm{ind}(D^+) = ?$ where $D^+ : \Omega_c^+(M_0/S^1) \longrightarrow \Omega_c^-(M_0/S^1)$.

Local description of the signature operator

Close to the fixed point set:

$$D = \gamma \left(\frac{\partial}{\partial r} + \bigstar \otimes A(r) \right) \Longrightarrow D^{+} = \frac{\partial}{\partial r} + A(r),$$

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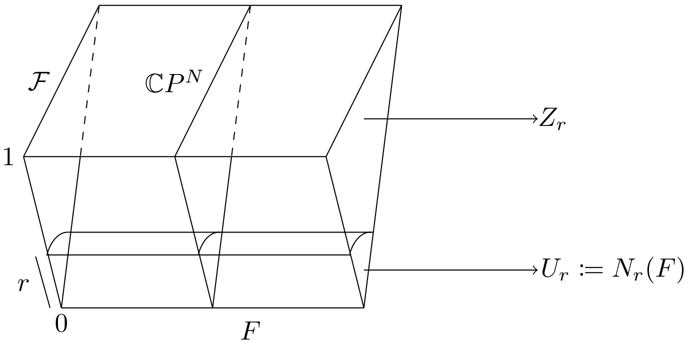
  - If N odd (Witt case) this is the case.
  - If N even (non-Witt case) $\Longrightarrow$ need boundary conditions.

Index formula for the Witt case (Brüning 09')



Use the Dirac-Systems formalism (Ballmann, Brüning & Carron, 08').

- In particular the gluing index formula:

$$\mathrm{ind}(D^+) = \mathrm{ind}\left(D^+_{Z_r, Q_<(A(r))(H)}\right) + \mathrm{ind}\left(D^+_{U_r, Q_\geq(A(r))(H)}\right).$$

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3. Finally,
   $$\lim_{r \to 0^+} \operatorname{ind}\left(D^+_{Z_r, Q_<(A_0(r))(H)}\right) + \frac{1}{2}\dim(\ker A_0(r)) = \sigma_{S^1}(M)$$

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Theorem (Brüning & Heintze, 79')

$$\begin{tikzcd} L^2(X,E)^G \arrow[r, "P|_{Dom(P)\cap L^2(X,E)^G}"] \arrow[d, "\Phi"'] & L^2(X,E)^G \arrow[d, "\Phi"] \\ L^2(X_0/G,F,h) \arrow[r, "T"] & L^2(X_0/G,F,h) \end{tikzcd}$$

Which self-adjoint operator to consider?

Main Idea: Push down an operator from M to M_0/S^1.

A first candidate is $D_M = d_M + d_M^\dagger$ acting on sections of $E \coloneqq \wedge T^* M$.

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- $F = \wedge T^*(M_0/S^1) \oplus \wedge T^*(M_0/S^1)$.
- $\omega \in \Omega_c(M_0)^{S^1} \Rightarrow \omega_0 + \omega_1 \wedge \chi$ where $\omega_0, \omega_1 \in \Omega_{\mathrm{bas},c}(M_0)$ and $\chi := \|V\|\alpha$ is the characteristic form.

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- $F = \wedge T^*(M_0/S^1) \oplus \wedge T^*(M_0/S^1)$.
- $\omega \in \Omega_c(M_0)^{S^1} \Rightarrow \omega_0 + \omega_1 \wedge \chi$ where $\omega_0, \omega_1 \in \Omega_{\text{bas},c}(M_0)$ and $\chi := \|V\|\alpha$ is the characteristic form.

Let $\kappa := -d\log(\|V\|) \in \Omega^1_{\text{bas}}(M_0)$ be the mean curvature form.
Using Rummler's formula $\varphi_0 := d\chi + \kappa \wedge \chi \in \Omega^2_{\text{bas}}(M_0)$ one verifies

$$T = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} \iota_{\bar{\kappa}^\sharp} & \varepsilon(\bar{\varphi}_0 \wedge) \\ \varepsilon(\bar{\varphi}_0 \wedge)^\dagger & -\bar{\kappa}\wedge \end{pmatrix}$$

- $\varepsilon := (-1)^j$ on j-forms.
- $\kappa =: \pi_{S^1}^* \bar{\kappa}$ and $\varphi_0 =: \pi_{S^1}^* \bar{\varphi}_0$ for $\pi_{S^1} : M_0 \longrightarrow M_0/S^1$.

Which self-adjoint operator to consider?

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We conjugate by multiplication by $U := h^{-1/2} = \|V\|^{-1/2}$,

$$U^{-1}TU = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\hat{c}(\bar{\kappa}) & \varepsilon(\bar{\varphi}_0 \wedge) \\ \varepsilon(\bar{\varphi}_0 \wedge)^\dagger & -\frac{1}{2}\hat{c}(\bar{\kappa}) \end{pmatrix}$$

where $\hat{c}(\bar{\kappa}) := \bar{\kappa} \wedge + \iota_{\bar{\kappa}^\sharp}$.

An induced operator Dirac-Schrödinger operator

$$\begin{tikzcd}
\Omega_c^{\mathrm{ev}}(M_0)^{S^1} \arrow[rr, "B := -c(\chi)d_M + d_M^{\dagger}c(\chi)"] & & \Omega_c^{\mathrm{ev}}(M_0)^{S^1} \\
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$$\mathscr{D} := D + \frac{1}{2}c(\bar{\kappa})\varepsilon.$$

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- The spectrum of the cone coefficient satisfies

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\operatorname{spec}\begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \cap (-1/2, 1/2) = \emptyset.
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Thus, $\mathscr{D}$ is essentially self-adjoint.

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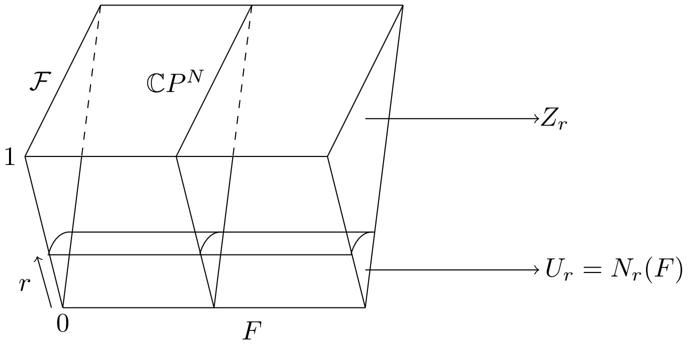
- The spectrum of the cone coefficient restricted to vertical harmonic forms is

$$2j - N \pm \frac{1}{2} \notin \left(-\frac{1}{2}, \frac{1}{2}\right),$$

  thus we see $\mathscr{D}$ is indeed essentially self-adjoint.

Index formula for the Witt case



- $\mathrm{ind}(\mathscr{D}^+) = \mathrm{ind}\left(\mathscr{D}^+_{Z_t, Q_<(A(r))(H)}\right) + \mathrm{ind}\left(\mathscr{D}^+_{U_t, Q_\geq(A(r))(H)}\right).$

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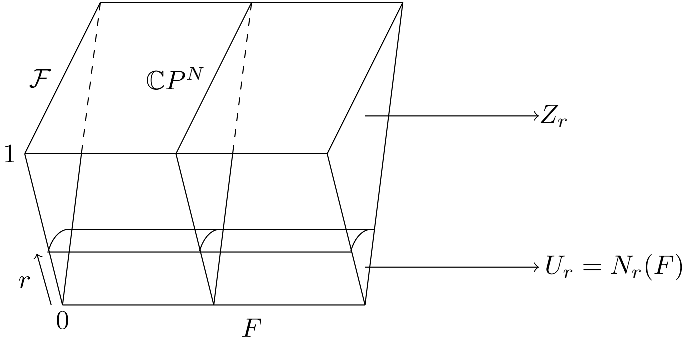

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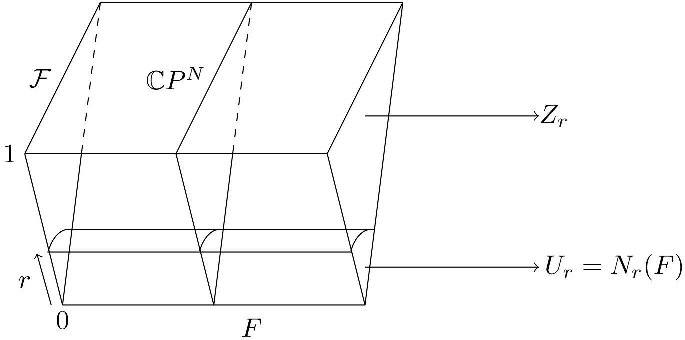

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- From a variation of Brüning's method we can prove

$$\lim_{r \to 0^+} \operatorname{ind}\left(\mathscr{D}^+_{Z_t, Q_<(A(r))(H)}\right) = \sigma_{S^1}(M).$$

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    - (Cheeger-Dai) $\sigma_{(2)}(T_\pi) = \tau_{\mathcal{F}}$. For non-Witt spaces there is a similar formula developed by Hunsicker & Mazzeo for the L^2-signature on the image $\mathcal{H}_{\mathrm{rel}} \longrightarrow \mathcal{H}_{\mathrm{abs}}$.

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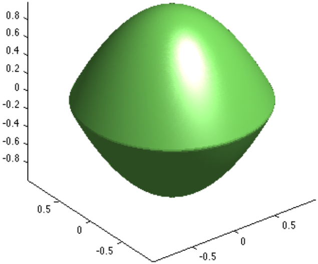
- Study the case of the spin Dirac operator (Studied in the Witt case by Albin and Gell-Redman).



Thank you!
Vielen Dank!
Gracias!
Gràcies!