INTRODUCTION TO THE CHERN CLASS

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Abstract. This document is presented as a final work for the seminar presented in the course *Index Theory*, given by Prof. Dr. Buening at Humbolt University, Berlin, Germany in the winter semester 2011/2012.

Contents

1. Characteristic Classes on Vector Bundles

1.1. **Curvature Form.** In this document we will work in the following framework: Let E be a (complex) dimensional vector bundle of rank m over a manifold M with projection $\pi : E \longrightarrow M$. We will assume that every manifold and every map is smooth unless otherwise stated.
Let $\varphi_U : U \subseteq M \longrightarrow U \times \mathbb{C}^m$ be local trivializations of the vector bundle E, and denote the transition functions by $g_{UV} : U \cap V \longrightarrow Gl(m, \mathbb{C})$.
Lets us denote by $\Gamma(M, E)$ the $C^\infty(M)$-module of sections on E. A connection on E is a linear map $\nabla : \Gamma(M, E) \longrightarrow \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, E)$, which satisfies the Leibniz identity $\nabla(fs) = df \otimes s + f\nabla s$ for all $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$. For a vector field $X \in \Gamma(M, TM)$ we denote the covariant derivative of s along X with respect to ∇ by $\nabla_X s \in \Gamma(M, E)$.

A set of local sections $\{s_1, ..., s_m\} \subseteq \Gamma(U, E)$ is said to be a local frame if for all $p \in U$, $\{s_1(p), ..., s_n(p)\}$ is a basis for the fiber $E_p := \pi^{-1}(p)$. For a

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vector field X we can expand in terms of these local sections

$$\nabla_X s_l = \sum_{j=1}^{m} \omega_U{}_l^j(X) s_j$$

from where we obtain a family of 1-forms $\omega_U{}_l^j$ on U which can be encoded in a single 1-form ω_U with values in the Lie algebra $\mathfrak{gl}(m, \mathbb{C})$. This differential form is called a connection1-form on U.

The linear map $R : \Gamma(M, E) \longrightarrow \Omega^2(M) \otimes_{C^\infty(M)} \Gamma(M, E)$ defined for vector fields X and Y by $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is called the curvature tensor of the connection ∇. In a local frame we can expand as

$$R(X, Y)(s_l) = \sum_{j=1}^{m} \Omega_U{}_l^j(X, Y) s_j,$$

from where we obtain a 2-form Ω_U on U with values in $\mathfrak{gl}(m, \mathbb{C})$, which is called a curvature 2-form for the connection on U.

The relation between ω_U and Ω_U is given by $d\omega_U = -\omega_U \wedge \omega_U + \Omega_U$, which actually means that

$$d\omega_U{}_l^j = -\sum_{r=1}^{m} \omega_U{}_r^j \wedge \omega_U{}_l^r + \Omega_U{}_l^j. \tag{1.1}$$

The transformation relations of these local forms are given by

$$\omega_V = g_{UV}^{-1} \omega_U g_{UV} + g_{UV}^{-1} dg_{UV} \tag{1.2}$$

$$\Omega_V = g_{UV}^{-1} \Omega_U g_{UV} \tag{1.3}$$

1.2. **Invariant Polynomials.** We shall give some results on the theory on invariant polynomials that will be essential for the study of characteristic classes.

Definition 1.1. A polynomial function

$$f : \mathfrak{gl}(m, \mathbb{C}) \to \mathbb{C}$$

is said to be **invariant** if it is invariant by similarity, in other words if $f(Q^{-1}AQ) = f(A)$ for all $Q \in GL(m, \mathbb{C})$. The most common examples are the trace and the determinant.

We will state two fundamental theorems on invariant polynomials:

Theorem 1.1. *Any symmetric polynomial in m variables can be uniquely expressed as*

$$\det(I + tA) = 1 + t\sigma_1(A) + ... + t^m \sigma_m(A) \quad \forall A \in \mathfrak{gl}(m, \mathbb{C}),$$

where each $\sigma_j(A)$ is an invariant polynomial on the eigenvalues of A.

Example 1.1. Let us compute

$$\det\left(I+\begin{pmatrix} a & b \\ c & d \end{pmatrix} t\right) = 1+(a+d)t+(ad-bd)t^2 = 1+tr(A)t+\det(A)t^2$$

Theorem 1.2. *The algebra $I_n(\mathbb{C})$ of invariant polynomials is a finitely generated $\mathbb{C}$-algebra generated by $\sigma_1, ..., \sigma_m$. If we define $s_j(A) = tr(A^j)$, then I_n is also generated by $s_1, ..., s_n$.*

$$I_m(\mathbb{C}) \cong \mathbb{C}[\sigma_1, ..., \sigma_m] \cong \mathbb{C}[s_1, ..., s_m]$$

With these results in mind we can describe the idea behind characteristic classes on vector bundles: Let f an invariant polynomial of degree k, by the transformation rule of equation (1.3) for the curvature form we ensure that $f(\Omega)$ is a globally defined $2k$-form since $f(\Omega_U) = f(\Omega_V)$ in $U \cap V$. Our goal is to show that $f(\Omega)$ is a closed form and therefore it defines a cohomology class in $H^{2k}_{dR}(M, \mathbb{C})$. Moreover, we will show that this class does not depend on the choice of the connection.

Proposition 1.1. *If $f \in I_n(\mathbb{C})$ is a polynomial of degree k, then $f(\Omega) \in \Omega^{2k}(M)$ is a closed form.*

Proof. We will need an expression which is called Bianchi's Identity: from equation (1.1) we find $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$. Form Theorem 1.2 it is clear that is enough to show that $ds_j(\Omega) = tr(d\Omega^j) = 0$. Using Leibniz rule and Bianch's identity we compute

$$d\Omega^j = \sum_{\alpha=1}^{j-1} \Omega^{\alpha-1} \wedge d\Omega \wedge \Omega^{j-\alpha} = \sum_{\alpha=1}^{j-1} \Omega^{\alpha} \wedge \omega \wedge \Omega^{j-\alpha} - \Omega^{\alpha-1} \wedge \omega \wedge \Omega^{j-\alpha+1}.$$

This is a telescopic sum, so we obtain $d\Omega^j = \Omega^j \wedge \omega - \omega \wedge \Omega^j$. Finally, using the fact that $tr(AB) = tr(BA)$ and that $\omega^i_\alpha \wedge \Omega^\alpha_r = \Omega^\alpha_r \wedge \omega^i_\alpha$ we conclude that $tr(\Omega^j \wedge \omega) = tr(\omega \wedge \Omega^j)$. □

Proposition 1.2. *For an invariant polynomial f of degree k, the de Rham cohomology class $[f(\Omega)] \in H^{2k}_{dR}(M, \mathbb{R})$ is independent of choice of the connection ∇.*

Proof. The idea of the proof is to use the fact that homotopic maps induce the same map in cohomology. Let ∇_0 y ∇_1 be two connections over E, with curvature forms Ω^0 and Ω^1 respectively. Let $p_1 : M \times \mathbb{R} \longrightarrow M$ the natural projection onto the first component. Consider the pullback connections $\tilde{\nabla}_0 = p_1^* \nabla_0$ and $\tilde{\nabla}_1 = p_1^* \nabla_1$ with curvature forms $\tilde{\Omega}_0$ and $\tilde{\Omega}_1$ over $p_1^* E$. For each $t \in [0, 1]$ define a connection on $p_1^* E$ by $\tilde{\nabla} = t\tilde{\nabla}_0 + (1-t)\tilde{\nabla}_1$ (it is important to take a convex linear combination) with curvature form $\tilde{\Omega}$.
If we denote, for $\epsilon = 0, 1$, the natural inclusion $\iota_\epsilon : M \longrightarrow M \times \mathbb{R}$ with $\iota_\epsilon(x) = (x, \epsilon)$, we see that $\iota_\epsilon^* \tilde{\Omega} = \Omega_\epsilon$ and moreover $\iota_\epsilon^* f(\tilde{\Omega}) = f(\Omega_\epsilon)$. Hence, since ι_0 and ι_1 are homotopic we conclude that

$$[f(\Omega_0)] = [\iota_0^* f(\tilde{\Omega})] = [\iota_1^* f(\tilde{\Omega})] = [f(\Omega_1)].$$

□

Definition 1.2. Let E be a vector bundle over M and f an invariant polynomial of degree k. Since the cohomology class $[f(\Omega)] \in H^{2k}_{dR}(M,\mathbb{C})$ does not depend on the connection we will denote it by $f(E)$ and call it the **characteristic class** of E corresponding to f.
In particular, the characteristic class corresponding to the polynomial

$$\left(\frac{-1}{2\pi i}\right)^k \sigma_k \in \mathrm{I}_m(\mathbb{C}) \tag{1.4}$$

is written $c_k(E) \in H^{2k}_{dR}(M,\mathbb{C})$ and it is called **Chern class** of degree k. The **total Chern class**, denoted by $c(E)$, can be written in terms of any curvature form on the vector bundle by

$$\left[\det\left(I - \frac{1}{2\pi i}\Omega\right)\right] = 1 + c_1(E) + c_2(E) + \cdots + c_m(E) \in H^*_{dR}(M,\mathbb{C}) \tag{1.5}$$

2. Clasification of Line Bundles

2.1. **Čech cohomolgy.** The goal of this section is to recall the definition of the Čech cohomology and to stay the image of the Chern class under the deRham isomorphism.

Definition 2.1. Let $\mathcal{U} = \{U_j\}_{j\in J}$ be a **contractible** open cover for M, that is, every non empty finite intersection $U_{j_0} \cap U_{j_2} \cdots \cap U_{j_k}$ is contractible. To any ordered set $j_0, j_1 \cdots j_k$ of $k+1$ distinct elements of the set J, such that $U_{j_0} \cap U_{j_2} \cdots \cap U_{j_k} \neq 222$ we assign a real number number $\alpha(j_0, j_1, \cdots, j_k)$, called a k**-cochain** with respect to $\mathcal{U}$, such that for every permutation $\sigma \in S_{k+1}$ we have

$$\alpha(j_{\sigma(0)}, j_{\sigma(1)}, \cdots, j_{\sigma(k)}) = \mathrm{sign}(\sigma)\alpha(j_0, j_1, \cdots, j_k) \tag{2.1}$$

The set of k-cochains $C^k(M;\mathcal{U})$ carries a natural vector space structure.

Remark 2.1. The existence of contractible covers can be seen as follows: introduce a Riemannian metric on M, then each point has a geodesically convex neighborhood and every intersection of two geodesically convex neighborhoods is still geodesically convex. (CITA)

Definition 2.2. We define a **coboundary operator** $\delta : C^k(M;\mathcal{U}) \longrightarrow C^{k+1}(M;\mathcal{U})$ by

$$\delta\alpha(j_0, j_1, \cdots, j_{k+1}) = \sum_{l=0}^{k} (-1)^l \alpha(j_0, j_1, \cdots j_{l-1}, \hat{j_l}, j_{l+1}, \cdots, j_{k+1}).$$

This definition ensures that δ satisfies $\delta^2 = 0$, and therefore we can define closed cochains, exact cochains and the **Čech cohomology group** of order k by $\check{H}^k(M,\mathcal{U})$ in the usual way. Although is not apparent form the definition, it can be shown that this cohomology group does not depend on the covering $\mathcal{U}$, but only on M, which is why we will denote it by $\check{H}^k(M,\mathbb{R})$.

The de Rham theorem [**?**] asserts that there is a natural isomorphism $\check{H}^k(M,\mathbb{R}) \cong H^k_{dR}(M,\mathbb{R})$. Let us discuss this isomorphism for the special case $k = 2$ Let $[\Omega] \in H^2_{dR}(M,\mathbb{R})$ and let $\mathcal{U} = \{U_j\}_{j\in J}$ be a contractible of open cover of M. Since U_j is contractible, by Poincaré lemma we have that locally $\omega|_{U_j} = d\theta_j$, moreover, since $U_{lj} = U_j \cap U_l$ is also contractible then $(\theta_l - \theta_j)|_{U_{lj}} = df_{jl}$ for some smooth real function f_{lj} on U_{lj}. Now, if we compute on U_{ljk},

$$df_{lj} + df_{jk} + df_{kl} = (\theta_j - \theta_l) + (\theta_k - \theta_j) + (\theta_l - \theta_k) = 0,$$

we se that $f_{lj} + f_{jk} + f_{kl}$ is a constant on U_{ljk} that we will denote by α_{ljk}. Let α the map defined by $\alpha(l,j,k) = \alpha_{ljk}$. It is easy to see that the map α satisfies the condition given by equation (2.1) and $\delta\alpha(i,j,k,l) = 0$, hence α defines a cohomology class on $\check{H}^2(M,\mathcal{U})$. Note the if $\tilde{\Omega} = \Omega + d\eta$, then $\tilde{\theta}_j = \theta_j + \eta$ on U_j. Hence, $(\tilde{\theta}_l - \tilde{\theta}_j)|_{U_{lj}} = (\theta_l - \theta_j)|_{U_{lj}} = df_{lj}$, so the map α is a well-defined map in cohomology.

2.2. **The Classification Theorem.** We will explore explicitly the de Rham isomorphism, discussed in section 2.3, applied to a Chern form $Tr(\Omega)$ of the first Chern class of a line bundle with connection. Let Ω be the curvature form associated to a compatible connection ∇ on an Hermitian line bundle. To find the image of a Chern form under the de Rham isomorphism we need to take

$$\omega = \left(\frac{1}{2\pi}\right) tr(\Omega) = \left(\frac{1}{2\pi}\right)\Omega$$

(the factor of the first Chern class changed as a result of the slight change on constants of the definition of the connection form we did in equation (**??**)) Therefore,

$$\left(\frac{1}{2\pi}\right)\Omega\Big|_{U_l} = \left(\frac{1}{2\pi}\right), d\Theta\Big|_{U_l}$$

where Θ is the connection form. Recall from Remark **??** that, on U_{jk}, we have

$$\left(\frac{1}{2\pi}\right)(\Theta_l - \Theta_j) = \left(\frac{1}{2\pi}\right) i\frac{dc_{jl}}{c_{jl}} = df_{lj},$$

hence, we can see that

$$\alpha_{jkl} = \frac{1}{2\pi i}(\log c_{jk} + \log c_{kl} + \log c_{lj}) \tag{2.2}$$

Notice that

$$\alpha_{jkl} = \frac{1}{2\pi i}\log(c_{jk}c_{kl}c_{lj}) = \frac{1}{2\pi i}\log(1) = \frac{1}{2\pi i}(2\pi i n) = n \qquad \text{for some} \quad n \in \mathbb{Z}$$

this shows that, under the de Rham isomorphism, the first Chern class lies in $\check{H}^2(M,\mathbb{Z})$.

As a conclusion we will show that the second Čech cohomology group, defined in section 2.3, classifies (up to isomorphism) the set of line bundles defined over a manifold M.

Theorem 2.1. *Let $L(M)$ the set of isomorphism classes of line bundles over M. Then there is a one-to-one correspondence between $L(M)$ and $\check{H}^2(M,\mathbb{Z})$.*

Proof. He have already seen that a family of transition functions $\{c_{jl}\}$ defines a cohomology class in $\check{H}^2(M,\mathbb{Z})$ given by equation (2.2). First of all we have to show that this cohomology class only depends on the isomorphism class of the line bundle defined by the transition functions. Consider two line bundles L_1, L_2 over M with transition functions given by $\{c^{(1)}_{jl}\}$ and $\{c^{(2)}_{jl}\}$, respectively. Then, it can be seen that L_1 and L_2 are equivalent if and only if there exists functions $\lambda_j : U_j \longrightarrow \mathbb{C}^\times$ such that $\lambda_j c^{(1)}_{jl}\lambda_l^{-1} = c^{(2)}_{jl}$ on $U_j \cap U_l$. If we compute

$$\begin{aligned}
\alpha^{(2)}_{jkl} &= \frac{1}{2\pi i}(\log c^{(2)}_{jk} + \log c^{(2)}_{kl} + \log c^{(2)}_{lj}) \\
&= \frac{1}{2\pi i}(\log(\lambda_j c^{(1)}_{jk}\lambda_k^{-1}) + \log(\lambda_k c^{(1)}_{kl}\lambda_l^{-1}) + \log(\lambda_l c^{(1)}_{lj}\lambda_j^{-1})) \\
&= \alpha^{(1)}_{jkl}\frac{1}{2\pi i} + \frac{\log(1)}{2\pi i},
\end{aligned}$$

we conclude that $[\alpha^{(1)}_{jkl}] = [\alpha^{(2)}_{jkl}]$. Therefore we can define a map $\kappa : L(M) \longrightarrow \check{H}^2(M,\mathbb{R})$ by $\kappa([L]) = [\alpha]$, which we will show is a bijection.
Now let $[\alpha_{jkl}] \in \check{H}^2(M,\mathbb{Z})$ and let $\mathcal{U} = \{U_j\}_{j\in J}$ be a contractible locally finite covering of M. Consider a partition of unity $\{b_j\}_{j\in J}$ subordinated to $\mathcal{U}$, and define smooth functions f_{jk} on on $U_j \cap U_k$ by $f_{jk} = \sum_{l\in J}\alpha_{jkl}b_l$, which are well defined since α_{jkl} is a cocycle. Notice that

$$f_{jk} + f_{kl} + f_{lj} = \alpha_{jkl} \in \mathbb{Z}$$

So, we can define $c_{jk} = \exp(2\pi i f_{jk})$ and it is clear that they satisfy the cocycle conditions, so they define a line bundle over M. Thus, we have shown that κ is surjective.
To show that κ is injective let L_1 and L_2 be two line bundles over M with transition functions $\{c^{(1)}_{jl}\}$ and $\{c^{(2)}_{jl}\}$ respectively, with respect to the contractible locally finite open cover described above, and suppose that $\kappa([L_1]) = \kappa([L_2])$. Define

$$v^{(r)}_{jk} = \frac{1}{2\pi i}\log c^{(r)}_{jk} \quad \text{for} \quad r = 1, 2.$$

Therefore is it easy to see that if we let $v_{jk} = v^{(1)}_{jk} - v^{(2)}_{jk}$ then $v_{jk} + v_{kl} + v_{lj} = 0$, where we have used that $\kappa([L_1]) = \kappa([L_2])$. Hence, we can define on U_j

$$\beta_j = \sum_{l\in J} v_{lj} b_l$$

we have that $\beta_j - \beta_k = f_{kj}$. Finally, if define $\lambda_j = \exp(2\pi i\beta_j)$, we have

$$\begin{aligned}\lambda_j c_{jl}^{(1)} \lambda_l^{-1} &= \exp(2\pi i(\beta_j - \beta_l))c_{jl}^{(1)} \\ &= \exp(2\pi i f_{lj}))c_{jl}^{(1)} \\ &= c_{lj}^{(1)}(c_{lj}^{(2)})^{-1}c_{jl}^{(1)} \\ &= c_{jl}^{(2)}\end{aligned}$$

so $[L_1] = [L_2]$. □

3. Dirac's Monopole

Consider a radial magnetic field of the form

$$B = \frac{\mu}{4\pi} vol_{S^2},$$

where $\mu = nh/q$, h is Plank's constant and $q > 0$ is the charge of the electron. We would like to ask ourselves, is there exists a magnetic potential $A \in \Omega^1(S^2)$ such that $dA = B$? Assume it does exist, then

$$0 \neq \mu = \int_{S^2} B = \int_{S^2} dA = \int_{\partial S^2} A = 0,$$

which is absurd (this is not surprising since $H_R^2(S^2) = \mathbb{R}$). Nevertheless, we cal always find local potentials, for example take

$$A^{(+)} = +\frac{\mu}{4\pi}(1 - \cos\theta)d\phi$$

Notice that

$$A^{(+)} - A^{(-)} = \frac{\mu}{2\pi} d\phi = \frac{h}{2\pi i q} e^{-in\phi} d(e^{in\phi})$$

Hence, we can interpret the local potentials as been local connection 1-forms of some line bundle with transition functions $\phi \longmapsto e^{in\phi}$

The Hamiltonian operator for this quantum system is given by

$$H = \frac{(p - qA)^2}{2m}$$

If we perform a gauge transformation $A \longmapsto A + d\lambda$, we require the wave function transforms as $\psi \longmapsto e^{2\pi i q\lambda/h}\psi$. In this particular case, the local expression of the local connection 1-forms differ by a potential $\lambda = nh\phi/2\pi q$, thus the wave function (which in lo longer a function, but a section) transforms as $\psi^{(+)} \longmapsto e^{in\phi}\psi^{(-)}$, therefore n must be an integer.

Now define the *true* connections by $\tilde{A}^{(+)} = (2\pi i q/h)A^{(+)}$ to obtain the curvature form $\tilde{\Omega} = (2\pi i q/h)B = (in/2)vol_{S^2}$, which allow us to compute the first Chern number of this bundle, namely,

$$C_1(\tilde{\Omega}) = \int_{S^2} \frac{-1}{4\pi} vol_{S^2} = -n \in \mathbb{Z}.$$

References

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