L^2-Cohomology and the Hodge Theorem

Juan Orduz

Berlin Mathematical School
Humboldt Universität zu Berlin
Geometrische Analysis und Spektraltheorie

BMS Student Conference 2016

Differential Forms

A **differential form** on a manifold M is something of the form

$$\omega(x) = f(x) dx^I.$$

- $f \in C^\infty(M, \mathbb{C})$ is a smooth function on M.
- $dx^I := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$,
  where $i_1 < i_2 < \cdots < i_k$ for $I = \{i_1, i_2, \cdots, i_k\} \subseteq \{1, 2, \cdots, n\}$.

Differential Forms

A **differential form** on a manifold M is something of the form

$$\omega(x) = f(x)dx^I.$$

- $f \in C^\infty(M, \mathbb{C})$ is a smooth function on M.
- $dx^I := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$,
  where $i_1 < i_2 < \cdots < i_k$ for $I = \{i_1, i_2, \cdots, i_k\} \subseteq \{1, 2, \cdots, n\}$.

Define the space of k**-forms** as $\Omega^k(M) := \{\omega = f dx^I \ : \ |I| = k\}$

- **Wedge Product:** $dx^I \wedge dx^J = (-1)^{|I||J|} dx^J \wedge dx^I$.
- **Exterior Derivative:** $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$,

$$d\left(f(x)dx^I\right) := \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x^j} dx^j \wedge dx^I.$$

Differential Forms

A **differential form** on a manifold M is something of the form

$$\omega(x) = f(x)dx^I.$$

- $f \in C^\infty(M, \mathbb{C})$ is a smooth function on M.
- $dx^I := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$,
  where $i_1 < i_2 < \cdots < i_k$ for $I = \{i_1, i_2, \cdots, i_k\} \subseteq \{1, 2, \cdots, n\}$.

Define the space of k**-forms** as $\Omega^k(M) := \{\omega = f dx^I \ : \ |I| = k\}$

- **Wedge Product:** $dx^I \wedge dx^J = (-1)^{|I||J|} dx^J \wedge dx^I$.
- **Exterior Derivative:** $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$,

$$d\left(f(x)dx^I\right) := \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x^j} dx^j \wedge dx^I.$$

- **Example:** $\omega = \cos(x^1)dx^{\{2,3\}} = \cos(x^1)dx^2 \wedge dx^3$, then

$$d\omega = -\sin(x^1)dx^1 \wedge dx^2 \wedge dx^3,$$
$$d(d\omega) = -\cos(x^1)dx^1 \wedge dx^1 \wedge dx^2 \wedge dx^3 = 0.$$

Berlin Mathematical School

The de Rham Complex

- The exterior derivative satisfies $d^2 = 0 \Rightarrow \mathrm{im}(d) \subseteq \ker(d)$.
- If we set $n := \dim M$ then we define the **de Rham Complex** of M as

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

- The **de Rham cohomology groups** are

$$H^*_{dR}(M) := \frac{\ker(d)}{\mathrm{im}(d)}.$$

- The **de Rham theorem** states that $H^*_{dR}(M) \cong H^*_{\mathrm{sing}}(M, \mathbb{R})$.

Berlin Mathematical School

Square integrable differential forms

Let M be a **closed** and **Riemannian** manifold. Consider the induced L^2-inner product on differential forms:

$$(\omega_1, \omega_2)_{L^2} := \int_M \langle \omega_1(x), \omega_2(x) \rangle dx.$$

Define the space of **square integrable forms** by $L^2(M) := \overline{\Omega(M)}^{L^2}$.

Square integrable differential forms

Let M be a **closed** and **Riemannian** manifold. Consider the induced L^2-inner product on differential forms:

$$(\omega_1, \omega_2)_{L^2} := \int_M \langle \omega_1(x), \omega_2(x) \rangle dx.$$

Define the space of **square integrable forms** by $L^2(M) := \overline{\Omega(M)}^{L^2}$.

Let $d^\dagger : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ be the **formal adjoint** of d, i.e

$$(d\omega_1, \omega_2)_{L^2} = (\omega_1, d^\dagger \omega_2)_{L^2} \quad \forall \omega_1, \omega_2 \in \Omega(M).$$

$$\begin{tikzcd} 0 \arrow[r] & \Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] \arrow[l, bend left, "d^\dagger"] & \cdots \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[r, "d"] & \Omega^n(M) \arrow[r] \arrow[l, bend left, "d^\dagger"] & 0. \end{tikzcd}$$

Hodge Theorem

The associated **Laplacian** is $\Delta := dd^\dagger + d^\dagger d$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(M) \arrow[r, "d"] \arrow[loop above, "\Delta"] & \Omega^1(M) \arrow[r, "d"] \arrow[loop above, "\Delta"] \arrow[l, bend left, "d^\dagger"] & \cdots \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[r, "d"] \arrow[loop above, "\Delta"] & \Omega^n(M) \arrow[r] \arrow[loop above, "\Delta"] \arrow[l, bend left, "d^\dagger"] & 0
\end{tikzcd}$$

Hodge Theorem

The associated **Laplacian** is $\Delta := dd^\dagger + d^\dagger d$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(M) \arrow[loop above, "\Delta"] \arrow[r, "d"] & \Omega^1(M) \arrow[loop above, "\Delta"] \arrow[r, "d"] \arrow[l, bend left, "d^\dagger"] & \cdots \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[loop above, "\Delta"] \arrow[r, "d"] & \Omega^n(M) \arrow[loop above, "\Delta"] \arrow[r] \arrow[l, bend left, "d^\dagger"] & 0
\end{tikzcd}$$

This positive second order differential operator is

- Elliptic. (Regularity)
- Essentially self-adjoint and discrete. (Spectral theorem)

The space of **harmonic forms** on M is

$$\mathcal{H}(M) := \ker(\Delta) \overset{!}{=} \{\omega \in \Omega(M) \mid d\omega = d^\dagger\omega = 0\} \quad (\dim \mathcal{H}(M) < \infty).$$

Hodge Theorem

The associated **Laplacian** is $\Delta := dd^\dagger + d^\dagger d$.

$$\begin{tikzcd} 0 \arrow[r] & \Omega^0(M) \arrow[loop above, "\Delta"] \arrow[r, "d"] & \Omega^1(M) \arrow[loop above, "\Delta"] \arrow[r, "d"] \arrow[l, bend left, "d^\dagger"] & \cdots \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[loop above, "\Delta"] \arrow[r, "d"] & \Omega^n(M) \arrow[loop above, "\Delta"] \arrow[r] \arrow[l, bend left, "d^\dagger"] & 0 \end{tikzcd}$$

This positive second order differential operator is

- Elliptic. (Regularity)
- Essentially self-adjoint and discrete. (Spectral theorem)

The space of **harmonic forms** on M is

$$\mathcal{H}(M) := \ker(\Delta) \overset{!}{=} \{\omega \in \Omega(M) \mid d\omega = d^\dagger\omega = 0\} \quad (\dim \mathcal{H}(M) < \infty).$$

We have a natural map

$$\begin{aligned} \mathcal{H}(M) &\longrightarrow H_{dR}(M) \\ \omega &\longmapsto [\omega] \end{aligned}$$

Hodge Theorem:
This map is a isomorphism.

Berlin Mathematical School

L^2-Cohomology

M = oriented Riemannian manifold of dimension n. Define

$$\Omega_c(M) := \{\omega \in \Omega(M) \,|\, \mathrm{supp}(\omega) \text{ is compact}\} \quad \text{and} \quad L^2(M) := \overline{\Omega_c(M)}^{L^2}.$$

In this case we define the formal adjoint $d^\dagger$ of d by

$$(d\omega_1, \omega_2)_{L^2} = (\omega_1, d^\dagger \omega_2)_{L^2} \quad \forall \omega_1, \omega_2 \in \Omega_c(M).$$

L^2-Cohomology

$M =$ oriented Riemannian manifold of dimension n. Define

$$\Omega_c(M) := \{\omega \in \Omega(M) \mid \mathrm{supp}(\omega) \text{ is compact}\} \quad \text{and} \quad L^2(M) := \overline{\Omega_c(M)}^{L^2}.$$

In this case we define the formal adjoint $d^\dagger$ of d by

$$(d\omega_1, \omega_2)_{L^2} = (\omega_1, d^\dagger \omega_2)_{L^2} \quad \forall \omega_1, \omega_2 \in \Omega_c(M).$$

Consider the exterior derivative d defined on

$$\Omega_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega \in L^2(M)\}.$$

This yields to a complex

$$0 \longrightarrow \Omega^0_{(2)}(M) \xrightarrow{d} \Omega^1_{(2)}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{(2)}(M) \longrightarrow 0$$

The associated cohomology groups $H^*_{(2)}(M)$ define L^2**-cohomology** of M, which we denote by $H^*_{(2)}(M)$.

Example

Consider $M = \mathbb{R}$ with the Euclidean metric. Then

$$H^k_{(2)}(\mathbb{R}) = \begin{cases} 0, & \text{if } k = 0 \\ \text{is infinite dimensional}, & \text{if } k = 1 \end{cases}$$

- Constant functions are not in $L^2(\mathbb{R})$.
- Let $\phi \in C_c(\mathbb{R}, \mathbb{C})$, then clearly $d(\phi dx) = 0$.
  If there exists $f \in C_c(\mathbb{R}, \mathbb{C})$ such that $df = \phi dx$ then

  $$\int_{\mathbb{R}} \phi(x) dx = \int_{\mathbb{R}} df(x) = 0,$$

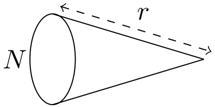
  since the support of f is compact.

Example: Conical singularity

Let N be a closed manifold of dimension n with Riemannian metric g_N. Define the **cone on** N by $C(N) := (0,1) \times N$ with metric Riemannian metric

$$g = dr^2 + r^2 g_N.$$

The L^2-cohomology groups are:



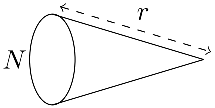
$$H^k_{(2)}(C(N)) = \begin{cases} H^k(N), \text{ if } k < \dfrac{n+1}{2} \\ 0, \text{ if } k \geq \dfrac{n+1}{2} \end{cases}$$

Example: Conical singularity

Let N be a closed manifold of dimension n with Riemannian metric g_N. Define the **cone on** N by $C(N) := (0,1) \times N$ with metric Riemannian metric

$$g = dr^2 + r^2 g_N.$$

The L^2-cohomology groups are:



$$H^k_{(2)}(C(N)) = \begin{cases} H^k(N), \text{ if } k < \dfrac{n+1}{2} \\ 0, \text{ if } k \geq \dfrac{n+1}{2} \end{cases}$$

For example: Let ω be an k-form on N and extend it trivially to $C(N)$, then

$$\int_{C(N)} |\omega|^2_g dx = \int_0^1 \int_N |\omega|^2_{g_N} r^{n-2k} dy dr,$$

thus $\omega \in L^2(C(N)) \iff n - 2k > -1$.

Berlin Mathematical School

Two closed extensions

Recall $\Omega_c(M) \subseteq \Omega_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega \in L^2(M)\}$.
We define two closed extensions of the exterior derivative:

- **Minimal extension:** $d_{\min}\omega = \beta$

$$\omega \in \mathrm{Dom}(d_{\min}) \Longleftrightarrow \exists\, (\omega_n)_n \subset \Omega_c(M) \text{ such that } \omega_n \longrightarrow \omega \\ \text{and } d\omega_n \longrightarrow \beta \text{ for some } \beta \in L^2(M).$$

- **Maximal extension** $d_{\max}\omega = \beta$

$$\omega \in \mathrm{Dom}(d_{\max}) \Longleftrightarrow \exists\, (\omega_n)_n \subset \Omega_{(2)}(M) \text{ such that } \omega_n \longrightarrow \omega \\ \text{and } d\omega_n \longrightarrow \beta \text{ for some } \beta \in L^2(M).$$

Clearly $d \subset d_{\min} \subset d_{\max}$.

Two closed extensions

Recall $\Omega_c(M) \subseteq \Omega_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega \in L^2(M)\}$.
We define two closed extensions of the exterior derivative:

- **Minimal extension:** $d_{\min}\omega = \beta$

$$\omega \in \mathrm{Dom}(d_{\min}) \Longleftrightarrow \exists\, (\omega_n)_n \subset \Omega_c(M) \text{ such that } \omega_n \longrightarrow \omega \text{ and } d\omega_n \longrightarrow \beta \text{ for some } \beta \in L^2(M).$$

- **Maximal extension** $d_{\max}\omega = \beta$

$$\omega \in \mathrm{Dom}(d_{\max}) \Longleftrightarrow \exists\, (\omega_n)_n \subset \Omega_{(2)}(M) \text{ such that } \omega_n \longrightarrow \omega \text{ and } d\omega_n \longrightarrow \beta \text{ for some } \beta \in L^2(M).$$

Clearly $d \subset d_{\min} \subset d_{\max}$.

- If we define $H_{(2),\#}(M) := \dfrac{\ker(d_{\max})}{\mathrm{im}(d_{\max})}$ then $H_{(2)}(M) \cong H_{(2),\#}(M)$.
- **Reduced L^2-cohomology:**

$$\bar{H}_{(2),\#}(M) := \frac{\ker(d_{\max})}{\overline{\mathrm{im}(d_{\max})}}.$$

Berlin Mathematical School

L^2-Harmonic Forms

We define the space of L^2**-harmonic forms** by

$$\mathcal{H}_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega = d^\dagger\omega = 0\}$$

• Is the inclusion $I : \mathcal{H}_{(2)}(M) \longrightarrow H_{(2)}(M)$ an isomorphism?
If it does we say that the **strong Hodge theorem holds**.

L^2-Harmonic Forms

We define the space of **L^2-harmonic forms** by

$$\mathcal{H}_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega = d^\dagger\omega = 0\}$$

- Is the inclusion $I : \mathcal{H}_{(2)}(M) \longrightarrow H_{(2)}(M)$ an isomorphism?

If it does we say that the **strong Hodge theorem holds**.

- The map I is *surjective* if $\mathrm{im}(d_{\max})$ is closed, i.e.

$$\overline{\mathrm{im}(d_{\max})} = \mathrm{im}(d_{\max}).$$

  In particular, this holds if $\dim(H^*_{(2)}(M)) < \infty$.

L^2-Harmonic Forms

We define the space of L^2**-harmonic forms** by

$$\mathcal{H}_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega = d^\dagger \omega = 0\}$$

- Is the inclusion $I : \mathcal{H}_{(2)}(M) \longrightarrow H_{(2)}(M)$ an isomorphism?

If it does we say that the **strong Hodge theorem holds**.

- The map I is *surjective* if $\operatorname{im}(d_{\max})$ is closed, i.e.

  $$\overline{\operatorname{im}(d_{\max})} = \operatorname{im}(d_{\max}).$$

  In particular, this holds if $\dim(H^*_{(2)}(M)) < \infty$.
- The map I is *injective* if **Stokes theorem holds in the** L^2 **sense** (L^2ST), i.e. $d_{\min} = d_{\max}$, equivalently

  $$(d_{\max}\omega_1, \omega_2) = (\omega_1, d^\dagger_{\max}\omega_2) \quad \forall \omega_1 \in \operatorname{Dom}(d_{\max}), \omega_2 \in \operatorname{Dom}(d^\dagger_{\max}).$$

L^2-Harmonic Forms

We define the space of L^2**-harmonic forms** by

$$\mathcal{H}_{(2)}(M) := \{\omega \in \Omega(M) \cap L^2(M) \mid d\omega = d^\dagger\omega = 0\}$$

• Is the inclusion $I : \mathcal{H}_{(2)}(M) \longrightarrow H_{(2)}(M)$ an isomorphism?
If it does we say that the **strong Hodge theorem holds**.

- The map I is *surjective* if $\mathrm{im}(d_{\max})$ is closed, i.e.

$$\overline{\mathrm{im}(d_{\max})} = \mathrm{im}(d_{\max}).$$

  In particular, this holds if $\dim(H^*_{(2)}(M)) < \infty$.

- The map I is *injective* if **Stokes theorem holds in the** L^2 **sense** (L^2ST), i.e. $d_{\min} = d_{\max}$, equivalently

$$(d_{\max}\omega_1, \omega_2) = (\omega_1, d^\dagger_{\max}\omega_2) \quad \forall \omega_1 \in \mathrm{Dom}(d_{\max}), \omega_2 \in \mathrm{Dom}(d^\dagger_{\max}).$$

Main idea: From the Kodira decomposition we obtain

$$H_{(2)}(M) = \underbrace{\ker(d_{\max}) \cap \ker(d^\dagger_{\min})}_{(L^2ST) \Rightarrow = \mathcal{H}_{(2)}(M)} \oplus \left(\frac{\overline{\mathrm{im}(d_{\max})}}{\mathrm{im}(d_{\max})}\right)$$

Berlin Mathematical School

Conclusions and Remarks

Main Message:

If the L^2-cohomology has finite dimension and (L^2ST) holds, then the strong Hodge theorem holds, i.e.

$$H_{(2)}(M) \cong \mathcal{H}_{(2)}(M).$$

Conclusions and Remarks

Main Message:

If the L^2-cohomology has finite dimension and (L^2ST) holds, then the strong Hodge theorem holds, i.e.

$$H_{(2)}(M) \cong \mathcal{H}_{(2)}(M).$$

Concerning the (L^2ST):

- Gaffney: M complete $\Longrightarrow$ (L^2ST) holds.

Conclusions and Remarks

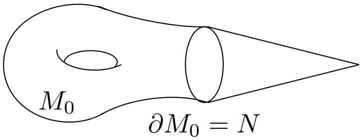
Main Message:

If the L^2-cohomology has finite dimension and (L^2ST) holds, then the strong Hodge theorem holds, i.e.

$$H_{(2)}(M) \cong \mathcal{H}_{(2)}(M).$$

Concerning the (L^2ST):

- Gaffney: M complete $\Longrightarrow$ (L^2ST) holds.
- For conical singularities $M = M_0 \cup C(N)$,



  Cheeger: (L^2ST) holds for M if:

  - (L^2ST) holds for N.
  - $H_{(2)}^{\dim N/2}(N) = 0$.

Conclusions and Remarks

- If (L^2ST) holds , Poincaré duality holds as well. Then L^2-signature of M is well defined in this case.

Conclusions and Remarks

- If (L^2ST) holds , Poincaré duality holds as well. Then L^2-signature of M is well defined in this case.
- There exists a Mayer-Vietoris sequences for L^2-cohomology.

Conclusions and Remarks

- If (L^2ST) holds , Poincaré duality holds as well. Then L^2-signature of M is well defined in this case.
- There exists a Mayer-Vietoris sequences for L^2-cohomology.
- The L^2-cohomology of singular spaces is intimately related to the **intersection cohomology** of Goresky-MacPherson $(I^pH^*(M))$. The parameter p is called a **perversity** and measures the failure of Poincaré duality on singular spaces.

Conclusions and Remarks

- If (L^2ST) holds , Poincaré duality holds as well. Then L^2-signature of M is well defined in this case.
- There exists a Mayer-Vietoris sequences for L^2-cohomology.
- The L^2-cohomology of singular spaces is intimately related to the **intersection cohomology** of Goresky-MacPherson $(I^pH^*(M))$. The parameter p is called a **perversity** and measures the failure of Poincaré duality on singular spaces.
- There are various extensions of L^2, for cohomology example, cohomology with coefficients or Dolbeault cohomology $(\bar{\partial})$ for complex manifolds.

Conclusions and Remarks

- If (L^2ST) holds , Poincaré duality holds as well. Then L^2-signature of M is well defined in this case.
- There exists a Mayer-Vietoris sequences for L^2-cohomology.
- The L^2-cohomology of singular spaces is intimately related to the **intersection cohomology** of Goresky-MacPherson $(I^pH^*(M))$. The parameter p is called a **perversity** and measures the failure of Poincaré duality on singular spaces.
- There are various extensions of L^2, for cohomology example, cohomology with coefficients or Dolbeault cohomology $(\bar{\partial})$ for complex manifolds.
- The appropriate setting to study all these notions are **Hilbert Complexes** [Brüning & Lesch].

$$0 \longrightarrow \mathscr{H}_0 \xrightarrow{d_0} \mathscr{H}_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} \mathscr{H}_{n-1} \xrightarrow{d_{n-1}} \mathscr{H}_n \longrightarrow 0$$

$\mathscr{H}_i =$ Hilbert space and $d_i : \mathrm{Dom}(d_i) \subseteq \mathscr{H}_i \longrightarrow \mathscr{H}_{i+1}$ are a closed operators such that $d_i \circ d_{i-1} = 0$.

THANK YOU!